HONOURS B.Sc., M.A. AND M.MATH. EXAMINATION

MATHEMATICS AND STATISTICS

Paper MT3600 Fundamentals of Pure Mathematics

January 2007

Time allowed : Two hours

Attempt all FOUR questions

[See over

1. Let X be a set, and let $\leq$ be a total order on X. We say that X is *dense* if for any two $x, y \in X$ with $x < y$ there exists $z \in X$ such that $x < z < y$. In what follows $\leq$ always denotes the usual ordering on rational numbers.

(i) For each of the following sets determine whether it is dense or not: (a) $\mathbb{Z}$ (integers); (b) $\mathbb{Q}$ (rationals); (c) $\{x \in \mathbb{Q} : 0 \leq x \leq 1\}$; (d) $\{x \in \mathbb{Q} : 0 \leq x \leq 1\} \cup \{x \in \mathbb{Q} : 2 \leq x \leq 3\}$. Justify your assertions. [4]

(ii) Let a and b be rational numbers satisfying $a < b$. To which of the following three sets does the number $\dfrac{2a+3b}{5}$ belong: $A = \{x \in \mathbb{Q} : x < a\}$, $B = \{x \in \mathbb{Q} : a < x < b\}$ or $C = \{x \in \mathbb{Q} : b < x\}$? Justify your answer. [1]

(iii) Consider the set

$$A = \{\frac{a}{5^n} : a, n \in \mathbb{Z},\ n \geq 0\}.$$

Is A dense? Justify your answer. [2]

(iv) Prove that $r = 1$ is the only positive rational number such that $r + \dfrac{1}{r}$ is an integer. [2]

(v) How many positive real numbers x are there such that $x + \dfrac{1}{x}$ is an integer? Your answer should be one of: 0, 1, 2, 3, …, countably infinite, or uncountably infinite, and you should justify it. [3]

2. A *Dedekind cut* is a set $\emptyset \neq A \subseteq \mathbb{Q}$ satisfying the following three conditions:

(C1) A is bounded above.

(C2) A has no maximum.

(C3) A is closed downwards, i.e. if $x \in A$ and $y \leq x$ then $y \in A$.

(i) For each of the following sets determine whether or not it is a Dedekind cut:

(a) $O = \{x \in \mathbb{Q} : x < 0\}$;

(b) $B = \{x \in \mathbb{Q} : x^2 < 2\}$;

(c) $C = \{x^2 : x \in \mathbb{Q},\ x < 2\}$;

(d) $D = \mathbb{Z}$.

You do not need to provide rigorous proofs of your assertions, but for each set that is not a Dedekind cut state at least one condition (C1), (C2), (C3) which it fails. [4]

(ii) Find a set $E \subseteq \mathbb{Q}$ which satisfies (C1) and (C3) but fails (C2). [1]

The *addition* of Dedekind cuts is defined as follows:

$$A + B = \{a + b \ : \ a \in A, \ b \in B\}.$$

In what follows you may assume without proof that if A and B are cuts then so is $A + B$.

(iii) Prove the commutative law for addition: $A + B = B + A$. [1]

(iv) Let A be a cut, and let t be an arbitrary positive rational number. Prove that there exists $a \in A$ such that $a + t \notin A$. Moreover, prove that a can be chosen so that $a + t$ is not the least upper bound for A. [3]

(v) Explain what is wrong with defining the *negative cut* corresponding to the cut A by

$$-A = \{-a \ : \ a \in A\}.$$ [1]

The *negative* of a cut A is actually defined as:

$$-A = \{-m \ : \ m \text{ is an upper bound, but not least, for } A\}.$$

In what follows you may use without proof the fact that if A is a cut then so is $-A$.

(vi) Prove that $A + (-A) = O = \{x \in \mathbb{Q} \ : \ x < 0\}$ for every cut A. You may wish to proceed as follows. To prove the direct inclusion, take $a \in A$ and $b \in -A$ and show that $a + b < 0$. For the converse inclusion first let $c < 0$ be arbitrary, then let $t = -c$, and then use (iv). [3]

3. (i) Express the number with the periodic decimal expansion $0.321212121\ldots$ in the form $\dfrac{m}{n}$, $m, n \in \mathbb{Z}$. [2]

(ii) Let r be the number $0.a_1a_2a_3\ldots$, where a_n is the last digit of n for every $n \in \mathbb{N}$. Is r rational or irrational? Justify your answer. [1]

(iii) Let s be the number $0.a_1a_2a_3\ldots$, where a_n is the first digit of n for every $n \in \mathbb{N}$. Is s rational or irrational? Justify your answer. [2]

Let $A \subseteq \mathbb{R}$ be the set of all real numbers with (infinite) decimal expansion of the form $0.d_1d_2d_3\ldots$ where $d_i \in \{2, 3\}$ for all $i \in \mathbb{N}$.

(iv) Prove that A contains infinitely many rational numbers, and also infinitely many irrational numbers. [2]

[See over

(v) Does A have a minimum (smallest element)? If so, what is it? Does A have a maximum (largest element)? [2]

(vi) Is A dense? Justify your answer. (For a reminder of the definition of density see Question 1.) [2]

(vii) Give an example of a set B of real numbers such that $B \subseteq [0,1]$, B is dense, countable and consists entirely of irrational numbers. Justify your assertions. [2]

4. (i) Prove that the open interval $(0,1)$ is uncountable. [3]

(ii) State (without proof) the Schröder–Bernstein Theorem. [1]

(iii) Using the Schröder–Bernstein Theorem, or otherwise, prove that the open interval $(0,1)$ and the open square $(0,1)\times(0,1)$ have the same cardinality. [3]

(iv) Determine which of the following three assertions is true:

(a) $\mathbb{C}$ (the set of all complex numbers) has cardinality less that $\mathbb{R}$ (the set of all real numbers);

(b) $\mathbb{R}$ has cardinality less than $\mathbb{C}$;

(c) $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality.

Justify your answer. You may assume without proof the existence of a bijection between the set of real numbers $\mathbb{R}$ and the open interval $(0,1)$. [2]

(v) The factorization of a natural number into a product of primes is unique up to reordering. That is, for all $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k \in \mathbb{N} \cup \{0\}$:

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k,$$

where $p_1, \ldots, p_k$ are distinct prime numbers.

Using this fact and the Schröder–Bernstein Theorem, or otherwise, prove that the set

$$\underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{n}$$

is countable for all $n \in \mathbb{N}$. [3]