HONOURS B.Sc., M.A. AND M.MATH. EXAMINATION

MATHEMATICS AND STATISTICS

Paper MT3600 Fundamentals of Pure Mathematics

September 2007

Time allowed : Two hours

Attempt all FOUR questions

[See over

1. Let X be a set, and let $\leq$ be a total order on X. We say that X is *dense* if for any two $x, y \in X$ with $x < y$ there exists $z \in X$ such that $x < z < y$. In what follows $\leq$ always denotes the usual ordering on rational numbers.

(i) For each of the following sets determine whether it is dense or not: (a) $\mathbb{Q}^+$ (positive rationals); (b) $\mathbb{N}$ (positive integers); (c) $\{x \in \mathbb{Q} \,:\, 3 \leq x \leq 4\}$; (d) $\{x \in \mathbb{Q} \,:\, 1 \leq x \leq 2\} \cup \{x \in \mathbb{Q} \,:\, 3 \leq x \leq 4\}$. Justify your assertions. [4]

(ii) Let a and b be rational numbers satisfying $a < b$. To which of the following three sets does the number $\dfrac{a+4b}{5}$ belong: $A = \{x \in \mathbb{Q} \,:\, x < a\}$, $B = \{x \in \mathbb{Q} \,:\, a < x < b\}$ or $C = \{x \in \mathbb{Q} \,:\, b < x\}$? Justify your answer. [1]

(iii) Consider the set

$$A = \{\frac{a}{5^n} \,:\, a, n \in \mathbb{Z},\ n \geq 0\}.$$

Is A dense? Justify your answer. [2]

(iv) Prove that $r = 1$ is the only positive rational number such that $r + \dfrac{1}{r}$ is an integer. [2]

(v) How many positive real numbers x are there such that $x + \dfrac{1}{x}$ is an integer? Your answer should be one of: 0, 1, 2, 3, ..., countably infinite, or uncountably infinite, and you should justify it. [3]

2. (i) Define what it means for a set $A \subseteq \mathbb{Q}$ to be a Dedekind cut. [2]

(ii) Give an example of a set $G \subseteq \mathbb{Q}$ that is a Dedekind cut, and an example of a set $H \subseteq \mathbb{Q}$ that is not a Dedekind cut. [2]

(iii) On the set $\mathbb{R}$ of all Dedekind cuts, define a relation $\leq$ as follows:

$$A \leq B \Leftrightarrow A \subseteq B.$$

Prove that $\leq$ is an order and that this order is total. [2]

(iv) Let $\mathcal{P}(\mathbb{Q})$ be the set of *all* subsets of $\mathbb{Q}$. Is it true that $\subseteq$ is a total order on $\mathcal{P}(\mathbb{Q})$? Justify your answer. [1]

(v) For a rational number r define the set

$$\overline{r} = \{x \in \mathbb{Q} \,:\, x < r\}.$$

Prove that $\overline{r}$ is a Dedekind cut. [2]

(vi) We proved in lectures that the set

$$A = \{x \in \mathbb{Q} \,:\, x^2 < 2\} \cup \{x \in \mathbb{Q} \,:\, x < 0\}$$

is a Dedekind cut. Prove that this Dedekind cut is not equal to any cut of the form $\overline{r}$, $r \in \mathbb{Q}$. [4]

3. (i) Express the number with the periodic decimal expansion $0.1353535\ldots$ in the form $\frac{m}{n}$, $m, n \in \mathbb{Z}$. [2]

(ii) Let r be the number $0.a_1a_2a_3\ldots$, where a_n is the penultimate digit of $n+10$ for every $n \in \mathbb{N}$. Is r rational or irrational? Justify your answer. [2]

(iii) Let s be the number $0.a_1a_2a_3\ldots$, where a_n is the first digit of n for every $n \in \mathbb{N}$. Is s rational or irrational? Justify your answer. [2]

Let $A \subseteq \mathbb{R}$ be the set of all real numbers with (infinite) decimal expansion of the form $0.d_1d_2d_3\ldots$ where $d_i \in \{1, 5\}$ for all $i \in \mathbb{N}$.

(iv) Prove that A contains infinitely many rational numbers, and also infinitely many irrational numbers. [2]

(v) Does A have a minimum (smallest element)? If so, what is it? Does A have a maximum (largest element)? [2]

(vi) Is A dense? Justify your answer. (For a reminder of the definition of density see Question 1.) [2]

(vii) Is the set A countable or uncountable? [1]

4. (i) Explain what is meant by saying that an infinite set is (a) *countable*; (b) *uncountable*. [2]

(ii) Give an example of a set S with a subset T such that S, T, and $S \setminus T$ are all countable. [2]

(iii) State (without proof) Cantor's Theorem. [2]

(iv) An infinite binary sequence is a sequence $x_1, x_2, x_3, \ldots$, where $x_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. Let B be the set of all infinite binary sequences. Prove that B has the same cardinality as
$mathcalP(\mathbb{N})$, the power set of the set of the natural numbers. [*Hint:* Suppose $x_1, x_2, x_3, \ldots$ is a binary sequence. Consider the set $\{n \in \mathbb{N} : x_n = 1\}$.] [4]

(v) Deduce that the set B is uncountable. [2]