Uncountable sets

Definition 20.1

An infinite set A is called *uncountable* or *uncountably infinite* if it is not countable: that is, if there is no bijection from $\mathbb{N}$ to A (i.e. $|\mathbb{N}| < |A|$).

We have not yet encountered any sets that we have proven to be uncountable.

$\mathbb{Q}$ and $\mathbb{R}$ are dense. So one might suspect that both $\mathbb{Q}$ and $\mathbb{R}$ are uncountable.

$(0, 1)$ is uncountable

Theorem 20.2

The interval $(0, 1)$ *is uncountable.*

$(0,1)$ is uncountable

Suppose $(0,1)$ is countable. Then there is a bijection $\phi : \mathbb{N} \to (0,1)$. Let a_{ij} be the jth decimal digit of $\phi(i)$. [No infinite tails of 9s]

n	$\phi(n)$							
1	0.	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
2	0.	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	$\cdots$
3	0.	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	$\cdots$
4	0.	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	$\cdots$
5	0.	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	$\cdots$
6	0.	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Read the diagonal to get $N_1 = 0.a_{11}a_{22}a_{33}\cdots$.

Let $N_2 = 0.b_1b_2b_3\cdots$ where each b_i is different from a_{ii}.

$N_2 \in (0,1)$, so $N_2 = \phi(k)$ for some $k \in \mathbb{N}$.

But the kth digit of N_2, b_k is different from the kth digit of $\phi(k)$, a_{kk}.

Contradiction.

$\mathbb{R}$

Theorem 20.3

$\mathbb{R}$ *is uncountable.*

Proof.

We want to show that $|\mathbb{N}| < |\mathbb{R}|$.
Recall that $|(-\pi/2, \pi/2)| = |\mathbb{R}|$ (use tan).
Now, $f : (0,1) \to (-\pi/2, \pi/2)$, where

$$x \mapsto \pi(x - \frac{1}{2})$$

is a bijection. Thus $|(0,1)| = |(-\pi/2, \pi/2)| = |\mathbb{R}|$.
But $(0,1)$ is uncountable, so $|\mathbb{R}| = |(0,1)| > |\mathbb{N}|$. ☐

Schröder–Bernstein Theorem

Theorem 21.1 (Schröder–Bernstein)

Let A and B be sets and let $f : A \to B$ and $g : B \to A$ be injections. Then $|A| = |B|$. More succinctly:

$$\big((|A| \leq |B|) \wedge (|B| \leq |A|)\big) \implies (|A| = |B|).$$

Schröder–Bernstein Theorem

We have injections $f : A \to B$ and $g : B \to A$.

If $f(a) = b$, call a the *parent* of b.
If $g(b) = c$, call b the *parent* of c.
(An element has at most one parent.)

Let $z \in A \cup B$. The *ancestral chain* of z is the unique longest sequence $z_0, z_1, \ldots$ such that $z_0 = z$ and z_{i+1} is the parent of z_i for each i.

If the ancestral chain for z is finite, then the index of its last element is the *depth* of z; otherwise z has *infinite depth*.

Schröder–Bernstein Theorem

$$\left.\begin{array}{c} A_e, B_e \\ A_o, B_o \\ A_\infty, B_\infty \end{array}\right\} \text{ are the subsets of } A, B \text{ of } \left\{\begin{array}{c} \text{even} \\ \text{odd} \\ \text{infinite} \end{array}\right\} \text{-depth elements.}$$

Notice that $f : A \to B$ maps:
A_e to B_o, A_o to B_e, and A_∞ to B_∞.
Similarly, $g : B \to A$ maps:
B_e to A_o, B_o to A_e, and B_∞ to A_∞.

An element of $A_o \cup B_o \cup A_\infty \cup B_\infty$ always has a parent; an element of $A_e \cup B_e$ may have no parent.

Schröder–Bernstein Theorem

Define $h : A \to B$ by

$$h(a) = \begin{cases} f(a) & \text{if } a \in A_e \cup A_\infty, \\ g^{-1}(a) & \text{if } a \in A_o. \end{cases}$$

This mapping is defined everywhere since $g^{-1}(a)$ exists and is unique for all $a \in A_o$.

Let $a_1, a_2 \in A$ with $h(a_1) = h(a_2)$.

1. If $h(a_1) = h(a_2)$ lies in B_e, then $a_1, a_2 \in A_o$. So $g^{-1}(a_1) = h(a_1) = h(a_2) = g^{-1}(a_2)$. So $a_1 = g(g^{-1}(a_1)) = g(g^{-1}(a_2)) = a_2$.
2. If $h(a_1) = h(a_2)$ lies in $B_o \cup B_\infty$, then $a_1, a_2 \in A_e \cup A_\infty$. So $f(a_1) = h(a_1) = h(a_2) = f(a_2)$. But f is injective, so $a_1 = a_2$.

Thus h is injective.

Schröder–Bernstein Theorem

$$h : A \to B, \qquad h(a) = \begin{cases} f(a) & \text{if } a \in A_e \cup A_\infty, \\ g^{-1}(a) & \text{if } a \in A_o. \end{cases}$$

Choose $b \in B$.

1. If $b \in B_e$, let $a = g(b)$. Then $h(a) = g^{-1}(g(b)) = b$.
2. If $b \in B_o \cup B_\infty$, then b has a parent $a \in A$ with $f(a) = b$. In fact, a must lie in $A_e \cup A_\infty$, so $h(a) = f(a) = b$.

Thus h is surjective.

Therefore h is a bijection from A to B and thus $|A| = |B|$.

Deferred proof

Theorem (Part 4 of Thm. 19.20)

If $|A| < |B|$ and $|B| < |C|$ then $|A| < |C|$.

Suppose $|A| < |B|$ and $|B| < |C|$. Let $f : A \to B$ and $g : B \to C$ be injections. Then $g \circ f : A \to C$ is an injection.

Suppose $h : A \to C$ is a bijection. Then $f \circ h^{-1} : C \to B$ is an injection. Applying the Schröder–Bernstein Theorem to $f \circ h^{-1}$ and g shows that $|B| = |C|$, which is a contradiction.

Hence $|A| < |C|$.

Countability of $\mathbb{N} \times \mathbb{N}$

Recall that $\mathbb{N} \times \mathbb{N}$ is countable.

$$\begin{array}{ccccccccc}
{}^{1}(1,1) & & {}^{2}(1,2) & & {}^{4}(1,3) & & {}^{7}(1,4) & & \dots \\
& \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
{}^{3}(2,1) & & {}^{5}(2,2) & & {}^{8}(2,3) & & {}^{12}(2,4) & & \dots \\
& \swarrow & & \swarrow & & \swarrow & & & \\
{}^{6}(3,1) & & {}^{9}(3,2) & & {}^{13}(3,3) & & {}^{18}(3,4) & & \dots \\
& \swarrow & & \swarrow & & & & & \\
{}^{10}(4,1) & & {}^{14}(4,2) & & {}^{19}(4,3) & & {}^{25}(4,4) & & \dots \\
\vdots & \swarrow & \vdots & & \vdots & & \vdots & &
\end{array}$$

Countability of $\mathbb{N} \times \mathbb{N}$

Formally, the required mapping is $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where

$$(x, y) \mapsto \left[\sum_{i=1}^{x+y-2} i \right] + x.$$

This is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. So $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Countability of $\mathbb{Q}$

Theorem 21.2

$\mathbb{Q}$ *is countable.*

Proof.

Define $g : \mathbb{Q} \to (\mathbb{Z} \times \mathbb{N})$ by $p/q \mapsto (p, q)$, where p and q are coprime.

g is an injection. Let h be a bijection from $\mathbb{Z}$ to $\mathbb{N}$. So the mapping $k : \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$, where

$$p/q \mapsto (h(p), q)$$

(p and q again being coprime), is an injection. So $|\mathbb{Q}| \leq |\mathbb{N}|$.

The inclusion $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ implies $|\mathbb{N}| \leq |\mathbb{Q}|$.

So, by the Schröder–Bernstein Theorem, $|\mathbb{Q}| = |\mathbb{N}|$. □