Properties of Cardinality

Theorem 21.1

Let A and B be disjoint sets, at least one of which is infinite. Then $|A \cup B| = \max(|A|, |B|)$.

- The formal proof of this result is beyond the scope of this course.
- Formally, the use of $\max(\ldots)$ is just an abbreviation for the following:

$$|A \cup B| = \begin{cases} |A| & \text{if } |B| \leq |A| \\ |B| & \text{if } |A| \leq |B|. \end{cases}$$

Cantor's Theorem

$\mathbb{P}A$, the *power set* of A is the set whose elements are the subsets of A:

$$\mathbb{P}A = \{B : B \subseteq A\}.$$

For example

$$\mathbb{P}\{1,2\} = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\}$$

$$\mathbb{P}\{1,2,3\} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{3,2\}, \{1,2,3\}\big\}$$

Theorem 21.2 (Cantor's Theorem)

For any set A, $|A| < |\mathbb{P}A|$.

Cantor's Theorem

Proof.

Note that $\mathbb{P}\emptyset = \{\emptyset\}$, so $|\emptyset| < |\mathbb{P}\emptyset|$. Assume henceforth that $A \neq \emptyset$.

The mapping $f : A \to \mathbb{P}A$, $a \mapsto \{a\}$ is an injection.

Suppose that $h : A \to \mathbb{P}A$ is a bijection.

Let $X = \{a \in A : a \notin h(a)\}$. Since h is surjective, there exists $b \in A$ with $h(b) = X$.

1. Suppose $b \in X$. Then, by the definition of X, $b \notin h(b) = X$, which is a contradiction.
2. Suppose $b \notin X$. Thus $b \notin h(b)$, and so, by the definition of X, $b \in X$, which is also a contradiction.

Therefore $|A| < |\mathbb{P}A|$. ☐

ZFC

- All of set theory can be built up from a small number of axioms.
- There are several such axiom systems, the most common of which is ZFC — Zermelo–Fraenkel Axioms (ZF) + Axiom of Choice (AC).

The Axiom of Choice is not universally accepted. AC may be expressed as follows:

> *If S is a set of disjoint nonempty sets, then there is a set R made up of exactly one element of each set in S.*

Alternatively: a unique representative can be chosen from each set in S and placed in a new set R.

Why use AC?

Certain intuitively correct results about set theory cannot be proven without the aid of AC:

- for any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$.
- if $A \neq \emptyset$, then there is an injection from A to B if and only if there is a surjection from B to A.

Why not use AC?

AC also has seemingly paradoxical consequences, such as the Banach–Tarski Theorem, which [loosely] asserts:

> *A solid sphere in three-dimensional Euclidean space can be split up into finitely many pieces that can then be reassembled to make two solid spheres, each the same size as the original.*

The Well-Ordering Principle and Zorn's Lemma

Definition 22.1

A *well-order* $<$ on a set A is a total order with the property that every subset of A has a minimum element.

The *well-ordering principle* (WO) asserts that every set admits a well-order.

Zorn's lemma (ZL) asserts the following:

> *If* $(X, <)$ *is a partially ordered set in which every totally ordered subset has an upper bound, then* X *has a maximum.*

The Well-Ordering Principle, Zorn’s Lemma, and the Axiom of Choice

Theorem 22.2

WO, ZL, and AC are all equivalent.

Proof.

See the exercises for a proof that WO implies AC. The remainder of the proof is beyond the scope of this course. ☐

Applying AC

Theorem 22.3

Every vector space has a basis.

(It is easy to show every *finite dimensional* vector space has a basis.)

Proof.

A proof can be found in the lecture notes. It is *not* examinable. □

A Question

Does there exist a set X such that $|\mathbb{N}| < |X| < |\mathbb{R}|$?

The Continuum Hypothesis

The conjecture:

> *There is no set X such that* $|\mathbb{N}| < |X| < |\mathbb{R}|$.

is called the *Continuum Hypothesis* (CH) and was originally formulated by Georg Cantor.

- Cantor could not prove or disprove CH.
- Still undecided in 1900 when David Hilbert put it first in his famous list of problems.

The Continuum Hypothesis

- In 1940, Kurt Gödel showed that one cannot *disprove* the Continuum Hypothesis in ZFC.
- In 1963, Paul Cohen showed that one cannot *prove* the Continuum Hypothesis in ZFC.

Therefore the Continuum Hypothesis is *independent* of ZFC. One can choose either to assume the Continuum Hypothesis or its negation.

The Generalized Continuum Hypothesis

The *Generalized Continuum Hypothesis* asserts that:

> *For any set Y, there does not exist a set X such that $|Y| < |X| < |\mathbb{P}Y|$.*