Fundamentals of Pure Mathematics

These slides can be downloaded from:

`www-groups.mcs.st-andrews.ac.uk/`
`~alanc/teaching/default.html`

$\mathbb{P}(\mathbb{N})$

Theorem 24.1

$|(0,1)| = |\mathbb{P}(\mathbb{N})|$. *(So $\mathbb{P}(\mathbb{N})$ is uncountable.)*

Proof.

Define $f : (0,1) \to \mathbb{P}(\mathbb{N})$ by

$$\underbrace{0.a_1a_2a_3\ldots}_{\text{binary expansion}} \mapsto \{i : a_i = 1\},$$

where $a_j \in \{0,1\}$.
f is an injection, so $|(0,1)| \leq |\mathbb{P}(\mathbb{N})|$.

$\mathbb{P}(\mathbb{N})$

Proof (cont.)

Define $g : \mathbb{P}(\mathbb{N}) \to (0, 1)$ as follows,

$$g(X) = \underbrace{0.a_1 0 a_2 0 a_3 \ldots}_{\text{binary expansion}} \quad (i \in X \implies a_i = 1, i \notin X \implies a_i = 0).$$

g is an injection.
So, by the Schröder–Bernstein theorem, $|(0, 1)| = |\mathbb{P}(\mathbb{N})|$. □

Why not $g(X) = 0.a_1 a_2 a_3 \ldots$? Because then

$$g(\{1\}) = 0.1000\ldots = 0.0111\ldots = g(\mathbb{N} - \{1\}).$$

(cf. $0.1000\ldots = 0.0999\ldots$ for decimal expansions.)

More countable sets

Proposition 24.2

Let P be the set of all subsets of $\mathbb{N}$ of size 2. Then P is countable.

Proof.

Clearly $|P| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Since $\mathbb{N}$ has the smallest cardinality of any infinite set, $|P| = |\mathbb{N}|$. □

More countable sets

Proposition 24.3

Let P_n be the set of all subsets of $\mathbb{N}$ of size n. For any $n \in \mathbb{N}$, P_n is countable.

Proof.

Clearly $|P_n| \leq |P_{n-1} \times \mathbb{N}|$. By induction, P_{n-1} is countable. So $|P_n| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. So $|P| = |\mathbb{N}|$. □

Finitary power set

Recall that $\mathbb{P}X$, the power set of X, is defined by

$$\mathbb{P}X = \{Y : Y \subseteq X\}.$$

The *finitary power set* of X, denoted $\mathbb{P}_F X$, is the set whose members are finite subsets of X

$$\mathbb{P}_F X = \{Y : Y \subseteq X, Y \text{ is finite}\}$$

Notice that

$$\mathbb{P}_F \mathbb{N} = \{\emptyset\} \cup \bigcup_{n \in \mathbb{N}} P_n.$$

Finitary power sets

Proposition 24.4

$\mathbb{P}_F(\mathbb{N})$ is countable.

Countable unions of countable sets

Theorem 24.5

If S_i is countable for each $i \in \mathbb{N}$, then

$$S = \bigcup_{i \in \mathbb{N}} S_i$$

is countable.

Proof.

Let $f_i : S_i \to \mathbb{N}$ be a bijection. Define $f : S \to \mathbb{N} \times \mathbb{N}$ by

$$x \mapsto (i, f_i(x))$$

where $x \in S_i$ but not in $S_1 \cup \cdots \cup S_{i-1}$.
This map is an injection, so $|S| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. So $|S| = |\mathbb{N}|$. □

Finitary power sets

Proof of Proposition 24.7.

$\mathbb{P}_F(\mathbb{N})$ is the union of the various countable sets P_n and the 1-element set $\{\emptyset\}$, so is itself countable. □

Fin

This is the end of the course, but we have a few matters to clear up:

- Surprise Egg Competition winners
- Feedback survey
- Next week — revision lectures on Monday and Tuesday (Jan 2007 paper)
- Next week — tutorial & examples class