PARTIAL SOLUTIONS FOR 'FUNDAMENTALS OF PURE MATHEMATICS' JANUARY 2007 EXAMINATION

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1. (a) (a) Not dense: $1, 2 \in \mathbb{Z}$ but there is no $x \in \mathbb{Z}$ with $1 < x < 2$. (a) Dense: if $x, y \in \mathbb{Q}$ with $x < y$ then $x < (x+y)/2 < y$ and $(x+y)/2 \in \mathbb{Q}$. (c) Dense: similar to (a), observe that $0 \leqslant (x + y)/2 \leqslant 1$. (d) Not dense: $1, 2$ lie in the set but there is no member of the set with $1 < x < 2$.

   (b) It lies in B since
$$a = \frac{5a}{5} = \frac{2a + 3a}{5} < \frac{2a + 3b}{5} < \frac{2b + 3b}{5} = \frac{5b}{5} = b.$$

   (c) A is dense. Pick $p/5^m, q/5^n \in A$ with $p/5^m < q/5^n$. Then
$$\frac{(2\cdot/5^m + 3q/5^n)}{5} = \frac{2 \cdot 5^n p + 3 \cdot 5^m q}{5^{m+n+1}} \in A,$$
   and by the previous part,
$$p/5^m < \frac{(2p/5^m + 3q/5^n)}{5} < q/5^n.$$

   (d) Suppose $r \in \mathbb{Q}^+$, $r + \frac{1}{r} \in \mathbb{Z}$. Let $r = p/q$ with p and q coprime (no common factor except 1) and $k \in \mathbb{Z}$ is such that
$$k = r + \frac{1}{r} = \frac{p}{q} + \frac{q}{p}.$$
   So $kpq = p^2 + q^2$. Therefore $p^2 = q(kp - q)$ and $q^2 = p(kq - p)$. Suppose t is a prime factor of q. Then it is a prime factor of p^2 and thus (by elementary facts about primes) a factor of p. This contradicts p and q being prime. So q has no prime factors, i.e. $q = 1$.
   Similar reasoning shows $p = 1$. So $r = p/q = 1$.

   (e) Let $k \in \mathbb{N}$ and suppose $r + 1/r = k$. Then $r^2 - kr + 1 = 0$. This equation has two solutions r_k, r'_k. List the solutions for all k:
$$r_1, r'_1 r_2, r'_2, r_3, r'_3, \ldots$$
   Thus the set of all positive reals r with $r + 1/r$ being an integer is countable.

2. (a) (a) O is a cut. (b) B is not a cut: fails (C3) because $1 \in B$ but $-2 \notin B$. (c) C is not a cut: fails (C1) since it contains $(-x)^2$ for large x. (d) D is not a cut: fails (C1) and (C3).

(b) Proceed as follows:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ &= \{b + a : b \in B, a \in A\} \\ &= B + A. \end{aligned}$$

(c) Pick $b \in A$. Consider $b + dt \in \mathbb{Q}$ for $d \in \mathbb{N} \cup \{0\}$. Since $t > 0$ and A is bounded above, $b + dt \notin A$ for some d. Let m be maximum with $b + mt \in A$ and $b + (m+1)t \notin A$. Set $a = b + mt$; then $a \in A$ and $a + t \notin A$.

Suppose that $a + t$ is the least upper bound of A. Then let $a' = a + t/2$. Then $a' \in A$ (since $a + t$ is the *least* upper bound) and $a' + t \notin A$ and $a' + t$ is not the least upper bound of A. So a with a' to complete the question.

(d) Many problems. Under this definition,

$$-O = -\{x \in \mathbb{Q} : x < 0\} = \{x \in \mathbb{Q} : x > 0\},$$

which isn't a cut — it fails conditions (C1) or (C3).

(e) Let $a \in A$, $b \in -A$. Then $b = -d$, where d is an upper bound, not least, for A. So $a < d$. So $a + b = a - d < 0$, so $a + b \in O$. Thus $A + (-A) \subseteq O$.

Let $c \in O$. Then $c < 0$. Let $t = -c$. Applying part (iv), there exist $a \in A$ with $a + t \notin A$ and $a + t$ not a least upper bound for A. Let $b = -(a+t) \in -A$. Then $a + b = a - (a + t) = -t = c$. So $c \in A + B$. Thus $O \subseteq A + B$.

3. (a) Proceed as follows:

$$\begin{aligned} &0.321212121\ldots \\ &= 0 + \frac{3}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \ldots \\ &= \frac{3}{10} + \frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \ldots \\ &= \frac{3}{10} + \frac{21}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right] \\ &= \frac{3}{10} + \frac{21}{10^3}\left[\frac{1}{1 - \frac{1}{100}}\right] \\ &= \frac{3}{10} + \frac{21}{10^3}\frac{100}{99} \\ &= \frac{3}{10} + \frac{21}{990} \\ &= \frac{312}{990}. \end{aligned}$$

(b) r is rational. The last digits of the natural numbers form a periodic sequence: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, and repeat. The numbers with periodic decimal expansions are precisely the rational numbers.

(c) s is not rational. The first digits of the natural numbers do not form a periodic sequence. For any k, the sequence of first digits of the natural numbers starting from 10^k begins with 10^k digits 1. So we can always find a string of 1s longer than any supposed period.

(d) There are many different ways to answer this question. For any $n \in \mathbb{N}$, A contains

$$x_n = 0.\underbrace{2\ldots2}_{n}3\underbrace{2\ldots2}_{n}3\underbrace{2\ldots2}_{n}3\ldots$$

The numbers x_n are all rational (they have periodic decimal expansion) and are all distinct. So A contains infinitely many rational numbers.

For any $n \in \mathbb{N}$, A contains

$$y_n = 0.\underbrace{2\ldots2}_{n}3\underbrace{2\ldots2}_{n+1}3\underbrace{2\ldots2}_{n+2}3\ldots$$

The numbers y_n are all irrational (they do not have periodic decimal expansion) and are all distinct. So A contains infinitely many irrational numbers.

(e) A has a minimum, namely $0.22222\ldots$, and a maximum, namely $0.33333\ldots$.

(f) A is not dense. For example $0.233333\ldots$ and $0.322222\ldots$ are both in A, but no real number between these two values lies in A.

(g) Let $p = \pi - 3$. Then $0 < p < 1/2$ and p is irrational. Let

$$B = \{p + x : x \in \mathbb{Q}, 0 < x < 1/2\}.$$

Then $B \subseteq [0, 1]$, B is dense since $\mathbb{Q}$ is dense, and each member of B is a sum of a rational and an irrational number and is therefore irrational.

4. (a) [This is the standard proof from the lectures.] Suppose $(0, 1)$ is countable. Then there is a bijection $\phi : \mathbb{N} \to (0, 1)$.

Let a_{ij} be the jth decimal digit of $\phi(i)$. [No infinite tails of 9s]

Let $N_1 = 0.a_{11}a_{22}a_{33}\cdots$. Let $N_2 = 0.b_1b_2b_3\cdots$ where each b_i is different from a_{ii}.

$N_2 \in (0, 1)$, so $N_2 = \phi(k)$ for some $k \in \mathbb{N}$.

But the kth digit of N_2, b_k is different from the kth digit of $\phi(k)$, a_{kk}. Contradiction.

(b) Let A and B be infinite sets. Let $f : A \to B$ and $g : B \to A$ be injections. Then $|A| = |B|$.

(c) Define $f : (0, 1) \times (0, 1) \to (0, 1)$ as follows:

$$(0.x_1x_2x_3\ldots, 0.y_1y_2y_3) \mapsto 0.x_1y_1x_2y_2x_3y_3\ldots.$$

This is an injection. Define $g : (0, 1) \to (0, 1) \times (0, 1)$ by

$$x \mapsto (x, 0.5).$$

Again, this is an injection. Apply the Schröder–Bernstein theorem to see that $|(0, 1)| = |(0, 1) \times (0, 1)|$.

(d) (c) $|\mathbb{R}| = |\mathbb{C}|$. Let $h : \mathbb{R} \to (0,1)$ be a bijection. By the previous part, there is a bijection $j : (0,1) \times (0,1) \to (0,1)$. Define $k : \mathbb{C} to \mathbb{R}$ by

$$x + yi \mapsto h^{-1}(j(h(x), h(y))).$$

Since h, j are bijections, so is k. So $|\mathbb{R}| = |\mathbb{C}|$.

(e) Define a mapping $f : \mathbb{N} \times \cdots \times \mathbb{N} \to \mathbb{N}$ by

$$(\alpha_1, \ldots, \alpha_n) \mapsto p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}.$$

This mapping is injective by the fact given in the question. The mapping $n \mapsto (n, 1, 1, \ldots, 1)$ is clearly an injection. Apply the Schröder–Bernstein theorem to see that $|\mathbb{N} \times \cdots \times \mathbb{N}| = |\mathbb{N}|$.