PARTIAL SOLUTIONS FOR 'FUNDAMENTALS OF PURE MATHEMATICS' SEPTEMBER 2007 RESIT EXAMINATION

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1. (a) (a) Dense: if $x, y \in \mathbb{Q}^+$ with $x < y$ then $x < (x+y)/2 < y$ and $(x+y)/2 \in \mathbb{Q}^+$. (b) Not dense: $1, 2 \in \mathbb{N}$ but there is no $x \in \mathbb{N}$ with $1 < x < 2$. (c) Dense: similar to (a), observe that $3 \leqslant (x+y)/2 \leqslant 4$. (d) Not dense $2, 3$ lie in the set but there is no member of the set with $2 < x < 3$.

   (b) It lies in B since
   $$a = \frac{5a}{5} = \frac{a+4a}{5} < \frac{a+4b}{5} < \frac{b+4b}{5} = \frac{5b}{5} = b.$$

   (c) A is dense. Pick $p/5^m, q/5^n \in A$ with $p/5^m < q/5^n$. Then
   $$\frac{(p/5^m + 4q/5^n)}{5} = \frac{5^n p + 4 \cdot 5^m q}{5^{m+n+1}} \in A,$$
   and by the previous part,
   $$p/5^m < \frac{(p/5^m + 4q/5^n)}{5} < q/5^n.$$

   (d) Suppose $r \in \mathbb{Q}^+$, $r + \frac{1}{r} \in \mathbb{Z}$. Let $r = p/q$ with p and q coprime (no common factor except 1) and $k \in \mathbb{Z}$ is such that
   $$k = r + \frac{1}{r} = \frac{p}{q} + \frac{q}{p}.$$
   So $kpq = p^2 + q^2$. Therefore $p^2 = q(kp - q)$ and $q^2 = p(kq - p)$. Suppose t is a prime factor of q. Then it is a prime factor of p^2 and thus (by elementary facts about primes) a factor of p. This contradicts p and q being prime. So q has no prime factors, i.e. $q = 1$.
   Similar reasoning shows $p = 1$. So $r = p/q = 1$.

   (e) Let $k \in \mathbb{N}$ and suppose $r + 1/r = k$. Then $r^2 - kr + 1 = 0$. This equation has two solutions r_k, r'_k. List the solutions for all k:
   $$r_1, r'_1 r_2, r'_2, r_3, r'_3, \ldots$$
   Thus the set of all positive reals r with $r + 1/r$ being an integer is countable.

2. (a) A nonempty subset A of $\mathbb{Q}$ is a Dedekind cut if it (a) is bounded above, (b) has no maximum, and (c) is closed downwards.

(b) Many examples, for instance $G = \{x \in \mathbb{Q} : x < 0\}$, $H = \{1\}$.

(c) The relation $\leqslant$ inherits reflexivity, anti-symmetry, and transitivity from the order $\subseteq$. Let A, B be cuts. We want to show that either $A \leqslant B$ or $B \leqslant A$. So assume $A \nleqslant B$. Then $A \not\subseteq B$. So there exists $a \in A \setminus B$. Let $b \in B$. Then $b < a$ (since $b > a$ would imply $a \in B$ since B is closed downwards). So $b \in A$ since A is closed downwards. So $b \in B \implies b \in A$. Hence $B \subseteq A$, i.e. $B \leqslant A$. So $\leqslant$ is a total order.

(d) The order is not total. For instance, the sets $\{0\}$ and $\{1\}$ are not comparable.

(e) $\bar{r}$ is bounded above by r. It has no maximum, since its least upper bound r does not lie in $\bar{r}$. It is clearly closed downwards. So $\bar{r}$ is a cut.

(f) Observe:

$$\begin{aligned} A &= \{x \in \mathbb{Q} : x^2 < 2\} \cup \{x \in \mathbb{Q} : x < 0\} \\ &= \{x \in \mathbb{Q} : x \geqslant 0, x^2 < 2\} \cup \{x \in \mathbb{Q} : x < 0\} \end{aligned}$$

Now suppose $A = \bar{r}$. Then

$$\begin{aligned} A^2 &= (\bar{r})^2 \\ \{x \in \mathbb{Q} : 0 < x < 2\} \cup \{x \in \mathbb{Q} : x < 0\} &= \{x \in \mathbb{Q} : x < r^2\} \\ \{x \in \mathbb{Q} : x < 2\} &= \{x \in \mathbb{Q} : x < r^2\}. \end{aligned}$$

So $r^2 = 2$, which is a contradiction since r is rational.

3. (a) Proceed as follows:

$$\begin{aligned} &0.1353535\ldots \\ &= 0 + \frac{1}{10} + \frac{3}{10^2} + \frac{5}{10^3} + \frac{3}{10^4} + \frac{5}{10^5} + \ldots \\ &= \frac{1}{10} + \frac{35}{10^3} + \frac{35}{10^5} + \frac{35}{10^7} + \ldots \\ &= \frac{1}{10} + \frac{35}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right] \\ &= \frac{1}{10} + \frac{35}{10^3}\left[\frac{1}{1 - \frac{1}{100}}\right] \\ &= \frac{1}{10} + \frac{35}{10^3}\frac{100}{99} \\ &= \frac{1}{10} + \frac{35}{990} \\ &= \frac{134}{990}. \end{aligned}$$

(b) r is rational. The second-last digits of the numbers $10, 11, 12, \ldots$ form a periodic sequence: ten digits 1, ten digits 2, ten digits 3, …ten digits 9, and repeat. The numbers with periodic decimal expansions are precisely the rational numbers.

(c) s is not rational. The first digits of the natural numbers do not form a periodic sequence. For any k, the sequence of first digits of the natural numbers starting from 10^k begins with 10^k digits 1. So we can always find a string of 1s longer than any supposed period.

(d) There are many different ways to answer this question. For any $n \in \mathbb{N}$, A contains

$$x_n = 0.\underbrace{1\dots1}_{n}5\underbrace{1\dots1}_{n}5\underbrace{1\dots1}_{n}5\dots$$

The numbers x_n are all rational (they have periodic decimal expansion) and are all distinct. So A contains infinitely many rational numbers.

For any $n \in \mathbb{N}$, A contains

$$y_n = 0.\underbrace{1\dots1}_{n}5\underbrace{1\dots1}_{n+1}5\underbrace{1\dots1}_{n+2}5\dots$$

The numbers y_n are all irrational (they do not have periodic decimal expansion) and are all distinct. So A contains infinitely many irrational numbers.

(e) A has a minimum, namely $0.11111\dots$, and a maximum, namely $0.55555\dots$.

(f) A is not dense. For example $0.155555\dots$ and $0.511111\dots$ are both in A, but no real number between these two values lies in A.

(g) A is uncountable.

4. (a) An infinite set X is countable if it has the same cardinality as $\mathbb{N}$: that is, if there is a bijection from X to $\mathbb{N}$. An infinite set X is uncountable if it has greater cardinality that $\mathbb{N}$: if there is no bijection from $\mathbb{N}$ to X.

(b) Many examples exist, for instance $S = \mathbb{N}$, $T = 2\mathbb{N} = \{2n : n \in \mathbb{N}\}$, $S \setminus T = \{2n - 1 : n \in \mathbb{N}\}$.

(c) For any set X, $|X| < |\mathbb{P}(X)|$.

(d) Define a map $f : B \to \mathbb{P}(\mathbb{N})$ by

$$x_1, x_2, x_3, \dots \mapsto \{n \in \mathbb{N} x_n = 1\}.$$

The mapping f is a bijection: its inverse is $g : \mathbb{P}(\mathbb{N}) \to B$, where

$$Y \mapsto y_1, y_2, y_3, \dots \text{ where } \begin{cases} y_i = 1 & \text{if } i \in Y \\ y_i = 0 & \text{if } i \in Y. \end{cases}$$

(Alternatively, prove that f is a bijection by showing it is injective and surjective.) So $|\mathbb{P}(\mathbb{N})| = |B|$.

(e) $|B| = |\mathbb{P}(\mathbb{N})| > |\mathbb{N}|$ by Cantor's Theorem and the previous part. So $|B| > |\mathbb{N}|$, i.e. B is uncountable.