GENERATING THE INFINITE SYMMETRIC GROUP USING A CLOSED SUBGROUP AND THE LEAST NUMBER OF OTHER ELEMENTS

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Abstract. Let S_∞ denote the symmetric group on the natural numbers $\mathbb{N}$. Then S_∞ is a Polish group with the topology inherited from $\mathbb{N}^\mathbb{N}$ with the product topology and the discrete topology on $\mathbb{N}$. Let $\mathfrak{d}$ denote the least cardinality of a dominating family for $\mathbb{N}^\mathbb{N}$ and let $\mathfrak{c}$ denote the continuum. Using theorems of Galvin, and Bergman and Shelah we prove that if G is any subgroup of S_∞ that is closed in the above topology and H is a subset of S_∞ with least cardinality such that $G \cup H$ generates S_∞, then, $|H| \in \{0, 1, \mathfrak{d}, \mathfrak{c}\}$.

The symmetric group S_∞ is a Polish group under the topology inherited from the product topology on $\mathbb{N}^\mathbb{N}$ with the discrete topology on $\mathbb{N}$; see [8, Section 9.B(7)] for further details. We will refer to subgroups G of S_∞ that are closed in this topology as closed subgroups. It is a well-known fact that the closed subgroups of S_∞ are precisely the automorphism groups of relational structures on $\mathbb{N}$; see, for example, [4, Theorem 5.8]. Such automorphism groups have been widely investigated; see for example [9] and the references therein.

The theorem of Bergman and Shelah from [3] on which our main theorem (Theorem 1.3) is based involves the following equivalence relation $\approx$ on subgroups of S_∞. Let G and H be (not necessarily closed) subgroups of S_∞. Then $G \approx H$ if there exists a countable $C \subseteq S_\infty$ such that the subgroup $\langle\, H, C\,\rangle$ generated by H and C equals $\langle\, G, C\,\rangle$.

Throughout, we write functions to the left of their argument and compose from right to left. The following two subgroups of S_∞ are representatives of two of the classes under $\approx$. Let A be a partition of $\mathbb{N}$ into sets $A_1, A_2, \ldots$ where $|A_i| = i$ for all $i \geq 1$ and let B be a partition of $\mathbb{N}$ into sets $B_0, B_1, \ldots$ with $|B_i| = 2$ for all $i \in \mathbb{N}$. Then define

$$\begin{aligned} H_\mathbb{N} &= \{\, f \in S_\infty \,:\, f(A_i) = A_i \text{ for all } i \geq 1 \,\} \\ H_2 &= \{\, f \in S_\infty \,:\, f(B_i) = B_i \text{ for all } i \in \mathbb{N} \,\}. \end{aligned}$$

It is straightforward to verify that $H_\mathbb{N}$ and H_2 are closed subgroups of S_∞. If G is a subgroup of S_∞ and $\Sigma \subseteq \mathbb{N}$, then the *pointwise stabilizer* of Σ in G is the subgroup $G_{(\Sigma)} = \{\, f \in G \,:\, f(\sigma) = \sigma \text{ for all } \sigma \in \Sigma\,\}$. We require the following slightly weaker version of the main theorem in Bergman and Shelah [3].

Theorem 1.1 (Bergman and Shelah [3]). *Let G be a closed subgroup of S_∞. Then exactly one of the following holds:*

(i) *$G_{(\Sigma)}$ has an infinite orbit for all finite $\Sigma \subseteq \mathbb{N}$ and $G \approx S_\infty$;*
(ii) *$G_{(\Sigma)}$ has only finite orbits for some finite $\Sigma \subseteq \mathbb{N}$, $G_{(\Gamma)}$ has orbits of unbounded length for all finite $\Gamma \subseteq \mathbb{N}$, and $G \approx H_\mathbb{N}$;*

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(iii) *$G_{(\Sigma)}$ has orbits of bounded length for some finite $\Sigma \subseteq \mathbb{N}$ and $G \approx H_2$ or $G \approx \{1_{\mathbb{N}}\}$.*

It is also shown in [3] that no two of the groups $\{1_{\mathbb{N}}\}$, H_2, $H_{\mathbb{N}}$, and S_∞ are equivalent under $\approx$.

Mesyan [13] considered the natural analogue of $\approx$ on the closed subsemigroups of the semigroup $\mathbb{N}^{\mathbb{N}}$. The situation is much more complicated in $\mathbb{N}^{\mathbb{N}}$, as in particular, there are infinitely many distinct equivalence classes in this setting. However, if the class of closed subsemigroups considered is restricted to endomorphism semigroups of bipartite graphs or partial orders on $\mathbb{N}$, then again a classification can be achieved. In particular, in [14] it is shown that such semigroups lie in precisely two equivalence classes under the analogue of $\approx$. In [12] Mesyan shows that certain closed subrings of the direct sum of $\aleph_0$ many copies of a simple ring fall into two distinct equivalence classes under the appropriate analogue of $\approx$.

Macpherson and Neumann [11, Theorem 1.1] showed that S_∞ is not the union of a countable chain of proper subgroups. Thus if G is any subgroup of S_∞ and $H \subseteq S_\infty$ with least cardinality such that $\langle G, H \rangle = S_\infty$, then $|H| \neq \aleph_0$. The following theorem of Galvin implies that, in fact, $|H| \leq 1$ or $|H| > \aleph_0$.

Theorem 1.2 (Theorem 5.8 in Galvin [6])**.** *Let G be subgroup of S_∞ such that there exists a countable $H \subseteq S_\infty$ with $\langle G, H \rangle = S_\infty$. Then there exists $f \in S_\infty$ such that $\langle G, f \rangle = S_\infty$.*

So, as Bergman and Shelah comment in [3], there is a gap between those subgroups of S_∞ over which it is 'easy' to generate S_∞ and those where it is 'difficult'. The aim of this paper is, in some sense, to make this comment more precise in the case of closed subgroups of S_∞.

If G is a subgroup of the symmetric group S_∞, then, for the sake of brevity, we will denote by $\operatorname{rank}(S_\infty : G)$ the least cardinality of a subset H of S_∞ such that the group generated by $G \cup H$ equals S_∞. Clearly, $\operatorname{rank}(S_\infty : G) \leq 1$ if and only if $S_\infty \approx G$. Also if $G \not\approx S_\infty$ and H is a subgroup of S_∞ such that $G \approx H$, then $\operatorname{rank}(S_\infty : G) = \operatorname{rank}(S_\infty : H) > \aleph_0$. The cardinal $\operatorname{rank}(S_\infty : G)$ will be referred to as the *relative rank of G in S_∞*. Relative ranks have been studied in the context of semigroups; see [5], [7], or [14] for further details.

A subset D of $\mathbb{N}^{\mathbb{N}}$ satisfying the following condition is called a *dominating family for $\mathbb{N}^{\mathbb{N}}$*:

> for all $f \in \mathbb{N}^{\mathbb{N}}$ there exists $g \in D$ such that $f(n) \leq g(n)$ for all $n \in \mathbb{N}$.

The least cardinality of a dominating family for $\mathbb{N}^{\mathbb{N}}$ is denoted by $\mathfrak{d}$ and the continuum is denoted by $\mathfrak{c}$. It is clear that $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$. If the continuum hypothesis holds, then $\aleph_1 = \mathfrak{d} = \mathfrak{c}$. However, the following theories are also consistent: ZFC + $(\mathfrak{d} = \aleph_1 < \mathfrak{c})$ and ZFC + $(\aleph_1 < \mathfrak{d} = \aleph_2 = \mathfrak{c})$; see [2] for further details.

The following is the main theorem of this paper.

Theorem 1.3. *Let G be a closed subgroup of S_∞. Then exactly one of the following holds:*

(i) *$G_{(\Sigma)}$ has an infinite orbit for all finite $\Sigma \subseteq \mathbb{N}$ and $\operatorname{rank}(S_\infty : G) \in \{0, 1\}$;*
(ii) *$G_{(\Sigma)}$ has only finite orbits for some finite $\Sigma \subseteq \mathbb{N}$, $G_{(\Gamma)}$ has orbits of unbounded length for all finite $\Gamma \subseteq \mathbb{N}$, and $\operatorname{rank}(S_\infty : G) = \mathfrak{d}$;*
(iii) *$G_{(\Sigma)}$ has orbits of bounded length for some finite subset Σ of $\mathbb{N}$ and $\operatorname{rank}(S_\infty : G) = \mathfrak{c}$.*

Proof. **(i).** By assumption and by Theorem 1.1, $G \approx S_\infty$ and so, by Theorem 1.2, it follows that $\operatorname{rank}(S_\infty : G) \in \{0, 1\}$.

(ii). Again by assumption and by Theorem 1.1, $G \approx H_\mathbb{N}$ and so $\operatorname{rank}(S_\infty : G) = \operatorname{rank}(S_\infty : H_\mathbb{N})$. Hence it suffices to prove that $\operatorname{rank}(S_\infty : H_\mathbb{N}) = \mathfrak{d}$.

We start by showing that $\operatorname{rank}(S_\infty : H_\mathbb{N}) \geq \mathfrak{d}$. By Tychonoff's Theorem [8, Proposition 4.1(vi)], $H_\mathbb{N}$ is compact. Let $U \subseteq S_\infty$ be such that $\langle H_\mathbb{N}, U \rangle = S_\infty$. Since $H_\mathbb{N} \not\approx S_\infty$, it follows that $|U| > \aleph_0$. The group $\langle H_\mathbb{N}, U \rangle$ is the union of sets $H_\mathbb{N} u_n H_\mathbb{N} u_{n-1} \cdots u_0 H_\mathbb{N}$ where $u_0, u_1, \ldots, u_n \in U \cup U^{-1}$ (where $U^{-1} = \{ u^{-1} : u \in U \}$). Since $H_\mathbb{N}$ is compact, right multiplication in any topological group is continuous, and the continuous image of a compact set is compact, it follows that each $H_\mathbb{N} u_i$ is compact. Hence, as the product of compact sets in a topological group is compact [1, Proposition 1.4.31], $H_\mathbb{N} u_n H_\mathbb{N} u_{n-1} \cdots u_0 H_\mathbb{N}$ is compact. In other words, $\langle H_\mathbb{N}, U \rangle$ is the union of $|U|$ compact sets. It is straightforward to prove that the minimum number of compact subsets covering $\mathbb{N}^\mathbb{N}$ is $\mathfrak{d}$. Hence, since S_∞ and $\mathbb{N}^\mathbb{N}$ are homeomorphic by [8, Theorem 7.7], it follows that the minimum number of compact subsets covering S_∞ is $\mathfrak{d}$. Thus $|U| \geq \mathfrak{d}$ and so $\operatorname{rank}(S_\infty : H_\mathbb{N}) \geq \mathfrak{d}$.

We will now prove that $\operatorname{rank}(S_\infty : H_\mathbb{N}) \leq \mathfrak{d}$. Let $M = \{m_0, m_1, \ldots\}$ be any subset of $\mathbb{N}$ with $|M| = |\mathbb{N} \setminus M| = |\mathbb{N}|$. Then, by Bergman and Shelah's Theorem 1.1(i), the pointwise stabilizer $S_{(\mathbb{N}\setminus M)}$ of $\mathbb{N} \setminus M$ in S_∞ satisfies $S_{(\mathbb{N}\setminus M)} \approx S_\infty$. Hence it suffices to prove that there exists $X \subseteq S_\infty$ with $|X| = \mathfrak{d}$ such that $S_{(\mathbb{N}\setminus M)} \subseteq \langle H_\mathbb{N}, X \rangle$.

Let D be any dominating family for $\mathbb{N}^\mathbb{N}$ with $|D| = \mathfrak{d}$. Replacing any $f \in D$ by $f' \in \mathbb{N}^\mathbb{N}$ defined recursively by setting $f'(0) = f(0) + 2$ and

$$f'(i) = \max\{f'(i-1) + 1, f(i)\}$$

for all $i > 0$, we may assume that D consists entirely of strictly increasing functions f' where $f'(i) \geq 2$ for all $i \in \mathbb{N}$. We denote by $T(f)$ the set of all involutions $g \in S_{(\mathbb{N}\setminus M)}$ (elements of order 2) such that $g(m_i) \in \{m_0, m_1, \ldots, m_{f(i)}\}$ for all $i \in \mathbb{N}$. Since D is a dominating family, $\bigcup_{f\in D} T(f)$ is the set of all involutions in $S_{(\mathbb{N}\setminus M)}$. The symmetric group S_∞ is generated by its involutions [6, Lemma 2.2], and so certainly $S_{(\mathbb{N}\setminus M)}$ is also generated by its involutions. Hence

$$\Big\langle \bigcup_{f\in D} T(f) \Big\rangle = S_{(\mathbb{N}\setminus M)}.$$

We will prove that for all $f \in D$ there exist $g_f, h_f \in S_\infty$ such that $T(f) \subseteq \langle H_\mathbb{N}, g_f, h_f \rangle$. Consequently,

$$S_{(\mathbb{N}\setminus M)} = \Big\langle \bigcup_{f\in D} T(f) \Big\rangle \subseteq \big\langle H_\mathbb{N} \cup \{ g_f, h_f \ : \ f \in D \} \big\rangle.$$

Thus $\operatorname{rank}(S_\infty : H_\mathbb{N}) \leq |D| = \mathfrak{d}$.

Let $f \in D$ be arbitrary and let N be any subset of $\mathbb{N}\setminus M$ such that $|N| = |\mathbb{N}\setminus[M \cup N]| = |\mathbb{N}|$. Partition N into sets $N_0, N_1, \ldots$ where $N_i = \{n_{i,0}, n_{i,1}, \ldots, n_{i,f(i)}\}$ and define $N_i^* = \{n_{r,i}, n_{r+1,i}, \ldots, n_{f(i),i}\}$ where $r = \min\{j \in \{0, 1, \ldots, f(i)\} : i \leq f(j)\}$. The subgroup $H_\mathbb{N}$ was defined as

$$H_\mathbb{N} = \{ g \in S_\infty \ : \ g(A_i) = A_i \text{ for all } i \geq 1 \}$$

where A is a partition of $\mathbb{N}$ into sets $A_1, A_2, \ldots$ satisfying $|A_i| = i$ for all $i \geq 1$. Hence, since $f(i) \geq 2$ for all $i \in \mathbb{N}$, it follows that $f(i) + 2 \leq |A_{2f(i)}| = 2f(i)$. Since the complements of $M \cup N$ and $\bigcup_{i\in\mathbb{N}} A_{2f(i)}$ in $\mathbb{N}$ are infinite, there exist $g_f, h_f \in S_\infty$ such that

$$g_f(N_i \cup \{m_i\}) \subseteq A_{2f(i)} \text{ and } h_f(N_i^* \cup \{m_i\}) \subseteq A_{2f(i)}$$

for all $i \in \mathbb{N}$.

Let $t \in T(f)$ be arbitrary and let $k \in S_\infty$ be such that $t(m_i) = m_{k(i)}$ for all $i \in \mathbb{N}$. Note that k is an involution since t is an involution. It follows from the definition of $T(f)$ that $k(i) \leq f(i)$ for all $i \in \mathbb{N}$ and so the element $n_{i,k(i)}$ exists for all $i \in \mathbb{N}$. Hence there exist involutions $p, q \in S_\infty$ such that p swaps m_i and $n_{i,k(i)}$ for all $i \in \mathbb{N}$ with $i < k(i)$, q swaps $m_{k(i)}$ and $n_{i,k(i)}$ for all $i \in \mathbb{N}$ with $i < k(i)$, and all other elements are fixed pointwise by p and q.

Since $m_i, n_{i,k(i)} \in N_i \cup \{m_i\}$, we have that $g_f(m_i), g_f(n_{i,k(i)}) \in A_{2f(i)}$ for all $i \in \mathbb{N}$. Hence the involution $g_f p g_f^{-1}$, which swaps $g_f(m_i)$ and $g_f(n_{i,k(i)})$ for all $i \in \mathbb{N}$ with $i < k(i)$, is an element of $H_\mathbb{N}$. Likewise, since $m_{k(i)}, n_{i,k(i)} \in N_{k(i)}^* \cup \{m_{k(i)}\}$, we have that $h_f q h_f^{-1} \in H_\mathbb{N}$. Therefore $p, q \in \langle\, H_\mathbb{N}, g_f, h_f \,\rangle$. Hence, as $p(m_{k(i)}) = m_{k(i)}$ and $q(m_i) = m_i$ for all $i \in \mathbb{N}$ with $i < k(i)$, it follows that $t = pqp \in \langle\, H_\mathbb{N}, g_f, h_f \,\rangle$, as required.

(iii). It suffices to prove that $\operatorname{rank}(S_\infty : H_2) = \mathfrak{c}$ and $\operatorname{rank}(S_\infty : \{1_\mathbb{N}\}) = \mathfrak{c}$. Clearly, $\operatorname{rank}(S_\infty : \{1_\mathbb{N}\}) = \mathfrak{c}$, as $\{1_\mathbb{N}\}$ is countable. We prove that $\operatorname{rank}(S_\infty : H_2) = \mathfrak{c}$ using the notion of relative ranks for semigroups. Replacing 'the group generated by' with 'the semigroup generated by' in the definition, it is clear what we mean by the relative rank of a subset of a semigroup. If $\operatorname{rank}(S_\infty : H_2) < \mathfrak{c}$, then there exists $U \subseteq S_\infty$ with $|U| < \mathfrak{c}$ and $\langle\, H_2, U \,\rangle = S_\infty$. Hence S_∞ equals the semigroup generated by H_2 and $U \cup U^{-1}$. By [7, Theorem 3.3], there exist $f, g \in \mathbb{N}^\mathbb{N}$ such that the semigroup generated by $S_\infty \cup \{f, g\}$ is $\mathbb{N}^\mathbb{N}$. Thus the semigroup relative rank of H_2 in $\mathbb{N}^\mathbb{N}$ is at most $|U \cup U^{-1} \cup \{f, g\}|$ and, in particular, it is strictly less than $\mathfrak{c}$. Hence, by the contrapositive, to prove that $\operatorname{rank}(S_\infty : H_2) = \mathfrak{c}$ it suffices to show that the semigroup relative rank of H_2 in $\mathbb{N}^\mathbb{N}$ is $\mathfrak{c}$.

Let A be a subset of $\mathbb{N}^\mathbb{N}$ such that H_2 and A generate $\mathbb{N}^\mathbb{N}$ as a semigroup. Seeking a contradiction assume that $|A| < \mathfrak{c}$. The semigroup $\mathbb{N}^\mathbb{N}$ is the union of the sets

$$B_{(a_0,a_1,\ldots,a_m)} = \{\, h_{m+1} a_m h_m \cdots h_1 a_0 h_0 \;:\; h_0, h_1, \ldots, h_{m+1} \in H_2 \,\}.$$

over all tuples $(a_0, a_1, \ldots, a_m) \in A^{m+1}$ and all $m \in \mathbb{N}$.

A subset F of $\mathbb{N}^\mathbb{N}$ is called *almost disjoint* if the set $\{\, i \in \mathbb{N} \;:\; f(i) = g(i) \,\}$ is finite for all $f, g \in F$. Let F be an almost disjoint subset of $\mathbb{N}^\mathbb{N}$ with $|F| = \mathfrak{c}$. The existence of such F is well-known, see, for example, [10, Theorem 1.3]. Since $|F| = \mathfrak{c}$ and $|A \cup A^2 \cup \cdots| < \mathfrak{c}$, there exist $n \in \mathbb{N}$ and $(b_0, b_1, \ldots, b_n) \in A^{n+1}$ such that $B_{(b_0,b_1,\ldots,b_n)} \cap F$ is infinite.

By the definition of H_2,

$$|\{\, f(i) \;:\; f \in B_{(b_0,b_1,\ldots,b_n)} \,\}| \leq 2^{n+2} \tag{1}$$

for all $i \in \mathbb{N}$. Let $N = 2^{n+2}$ and let $f_0, f_1, \ldots, f_N$ be distinct elements of $B_{(b_0,b_1,\ldots,b_n)} \cap F$. Then, since F is a family of almost disjoint functions, there exists $i \in \mathbb{N}$ such that $f_0(i), f_1(i), \ldots, f_N(i)$ are distinct, contradicting (1). □

We do not know a subgroup of S_∞ or a subsemigroup of $\mathbb{N}^\mathbb{N}$ with an uncountable relative rank not equal to $\mathfrak{d}$ or $\mathfrak{c}$. So, we ask: does there exist a subgroup G of S_∞ or a subsemigroup S of $\mathbb{N}^\mathbb{N}$ such that $\operatorname{rank}(S_\infty : G), \operatorname{rank}(\mathbb{N}^\mathbb{N} : S) \notin \{0, 1, 2, \mathfrak{d}, \mathfrak{c}\}$?

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