APPROXIMATION OF AUTOMORPHISMS OF THE RATIONALS AND THE RANDOM GRAPH

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ABSTRACT. Let G be the group of order-preserving automorphisms of the rationals $\mathbb{Q}$, or the group of colour-preserving automorphisms of the $\mathcal{C}$-coloured random graph $R_{\mathcal{C}}$. We show that given any non-identity $f \in G$, there exists $g \in G$ such that every automorphism in G is the limit of a sequence of automorphisms generated by f and g. Moreover, if, in some sense, f has no finite structure, then g can be chosen with a great deal of flexibility.

1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREMS

In this paper we consider finitely generated dense subgroups of the following Polish groups: $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ the order-preserving automorphisms of the rationals $\mathbb{Q}$ and $\mathrm{Aut}(R_{\mathcal{C}})$ the colour-preserving automorphisms of the $\mathcal{C}$-coloured random graph $R_{\mathcal{C}}$. We say that a Polish group G with a dense 2-generated subgroup is *topologically* 2*-generated*. There is an extensive list of Polish groups that are topologically 2-generated. For example[1], the symmetric group S_∞ on the natural numbers; the automorphism group $\mathrm{Aut}(R_{\mathcal{C}})$ of the $\mathcal{C}$-coloured random graph [12]; the group of homeomorphisms $H(2^{\mathbb{N}})$ of the Cantor space (Kechris and Rosendal [9, Theorem 2.10]); the automorphism group $\mathrm{Aut}(X, \mu)$ of a standard measure space (X, μ) (Grzaslewicz [6] and Prasad [13]); the group of isometries of the Urysohn space (Solecki [15]). If G is any of the groups listed above, then G satisfies a stronger property: there exist $f, g \in G$ such that $\{\ g^{-n} f g^n\ :\ n \in \mathbb{Z}\ \}$ is dense in G. Such a group G is said to have a *cyclically dense* conjugacy class. If G has a cyclically dense conjugacy class, then clearly it is both topologically 2-generated and has a dense conjugacy class.

Polish groups in general, and automorphism groups of Fraïssé limits in particular, with dense or comeagre conjugacy classes have been extensively studied. Several of the groups given above can be viewed as automorphism groups of Fraïssé limits: $(\mathbb{N}, =)$ is the Fraïssé limit of the class of finite sets; $(\mathbb{Q}, \leqslant)$ is the Fraïssé limit of the class of finite linear orders; $R_{\mathcal{C}}$ is the Fraïssé limit of the class of finite $\mathcal{C}$-coloured graphs; $H(2^{\mathbb{N}})$ is isomorphic to $\mathrm{Aut}(B_\infty)$ where B_∞ is the countable atomless Boolean algebra, i.e. the Fraïssé limit of the class of finite Boolean algebras. If G is a Polish group with a comeagre conjugacy class, then G is said to have

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[1]Although some of these examples are recent, the topic of finite topological generation can be traced back to the 1934 paper of Schreier and Ulam [14] where they showed that the continuous functions on $[0,1]^n$, $n \geqslant 1$, with the compact-open topology are topologically 5-generated.

generic elements. The significance of generic elements is that they utilize and interrelate the topological and algebraic structures of the group. The study of generic automorphisms was initiated by Truss [17] and Lascar [11]. In [17], Truss proved that $\mathrm{Aut}(R_{\mathcal{C}})$ and $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ have comeagre conjugacy classes and described their elements explicitly. Other examples of groups with generic elements include: the random poset (Kuske and Truss [10]), many ω-stable, $\aleph_0$-categorical structures (Hodges, Hodkinson, Lascar, and Shelah [7]); the free group on countably many generators (Bryant and Evans [2]); the isometry group of the rational Urysohn space (Solecki [15]); the group of Haar measure preserving homeomorphisms of the Cantor space (Kechris and Rosendal [9]); the group of Lipschitz homeomorphisms of the Baire space (Kechris and Rosendal [9]); the group of homeomorphisms of the Cantor space (Akin, Glasner, and Weiss [1], Kechris and Rosendal [9]). Let K be a Fraïssé limit and let $\mathrm{Aut}(K)$ be its automorphism group. Kechris and Rosendal [9] showed that the existence of a dense conjugacy class in $\mathrm{Aut}(K)$ is equivalent to the systems of isomorphisms between finite substructures of K having the joint embedding property. They also showed that the existence of generic automorphisms of K is equivalent to the systems of isomorphisms between finite substructures of K having the joint embedding property and the weak amalgamation property.

Prasad [13] proved that

$$\{\ (f, g) \in \mathrm{Aut}(X, \mu)^2\ :\ f, g \text{ topologically generate } \mathrm{Aut}(X, \mu)\ \}$$

is comeagre in $\mathrm{Aut}(X, \mu)^2$ where (X, μ) is a standard measure space. Let K be a Fraïssé limit and let $G = \mathrm{Aut}(K)$ be its automorphism group. Then, as pointed out in Kechris and Rosendal [9, p315], the set

$$D = \{\ (f, g) \in G^2\ :\ f, g \text{ topologically generate } G\ \}$$

is G_δ and invariant under conjugation by elements of G. The group G acts on G^2 by conjugation $g \cdot (h_1, h_2) = (g^{-1}h_1g, g^{-1}h_2g)$. It follows that if G has a dense orbit on G^2 under this action, then D is either comeagre or nowhere dense in G^2. In fact, D is not dense in G^2 whenever G is non-trivial and so D is nowhere dense if G has a dense orbit on G^2 under the action given above. In particular, D is nowhere dense when G is either of the groups $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ or $\mathrm{Aut}(R_{\mathcal{C}})$. Although D is in general nowhere dense, it is natural to ask for which $f \in G$ the set

$$\mathcal{D}_f = \{\ g \in G\ :\ f, g \text{ topologically generate } G\ \}.$$

is non-empty, or even comeagre in G. Alternatively, does there exist a G_δ subset H of G such that

$$\{\ (f, g) \in H^2\ :\ \langle\ f, g\ \rangle \text{ is dense in } G\ \}$$

is comeagre in H^2.

In [3] we answered these questions when $G = S_\infty$. We showed that given a non-identity $f \in S_\infty$, there exists $g \in S_\infty$ such that f and g generate S_∞ topologically. That is, the set $\mathcal{D}_f$ is non-empty for all $f \in S_\infty \setminus \{1_{\mathbb{N}}\}$. Let $\mathcal{I}$ and $\mathcal{S}$ denote the subsets of S_∞ consisting of those permutations without finite cycles and those with only one cycle, respectively. We showed that if f has infinite support (the set $\{\, i \in \mathbb{N} : f(i) \neq i \,\}$ is infinite), then $\mathcal{D}_f \cap \mathcal{S}$ is comeagre in $\mathcal{S}$ and $\mathcal{D}_f \cap \mathcal{I}$ is comeagre in $\mathcal{I}$. Furthermore, if $f \in \mathcal{I}$, then $\mathcal{D}_f$ is comeagre in S_∞. Reproducing a theorem of Dixon [4], as a corollary of these results and the Kuratowski-Ulam Theorem, we showed that $\{ (f, g) \in \mathcal{I}^2 : f, g \text{ topologically generate } S_\infty \}$ and $\{ (f, g) \in \mathcal{S}^2 : f, g \text{ topologically generate } S_\infty \}$ are comeagre in $\mathcal{I}^2$ and $\mathcal{S}^2$, respectively.

This paper is a continuation of our earlier work. Here we give a detailed analysis of pairs of elements that topologically generate the groups $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ and $\mathrm{Aut}(R_\mathcal{C})$. In the remainder of this section, we give some preliminary information and state our main theorems. More detailed background material, that will be required in the proofs of the main theorems, is given in Section 2. The proofs of the main theorems relating to $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ are in Section 3 and those relating to $\mathrm{Aut}(R_\mathcal{C})$ are in Section 4.

1.1. **The rationals.** Let $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be an arbitrary non-identity element. Then we show in Theorem 1.1 that in some cases there exist, not only one, but many $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ such that f and g topologically generate $\mathrm{Aut}(\mathbb{Q}, \leqslant)$. Moreover, we are relatively free to specify the cycle structure of g. Before we can state this theorem we require the following standard notions relating to $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ and $\mathrm{Aut}(R_\mathcal{C})$; more detailed information about these groups can be found in Section 2.

Let $G = \mathrm{Aut}(\mathbb{Q}, \leqslant)$ or $G = \mathrm{Aut}(R_\mathcal{C})$ and let $f \in G$ be arbitrary. Then we define

$$\mathcal{D}_f = \{ g \in G : f, g \text{ topologically generate } G \}.$$

If $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$, then there exists a unique order-preserving homeomorphism $\tilde{f}$ of $\mathbb{R}$ extending f. The set $\mathrm{fix}(\tilde{f}) = \{ \alpha \in \mathbb{R} : \tilde{f}(\alpha) = \alpha \}$ is a closed subset of $\mathbb{R}$. Conversely, if U is a closed subset of $\mathbb{R}$, then there exists $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ such that $U = \mathrm{fix}(\tilde{f})$. If U is a closed subset of $\mathbb{R}$, then a maximal open interval in $\mathbb{R} \setminus U$ is called a *component* of $\mathbb{R} \setminus U$. If U is a closed subset of $\mathbb{R}$, then define

$$\mathcal{F}_U = \{ f \in \mathrm{Aut}(\mathbb{Q}, \leqslant) : \mathrm{fix}(\tilde{f}) = U \}.$$

It is straightforward to see that $\mathcal{F}_U$ is a G_δ subset of $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ and so $\mathcal{F}_U$ is a Polish space (but not a group).

Theorem 1.1. *Let U and V be disjoint closed subsets of $\mathbb{R}$ such that one of the following holds:*

(i) *all the components of $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are bounded;*
(ii) *$\mathbb{R} \setminus U$ has an unbounded component and V is bounded.*

Then $\mathcal{D}_f \cap \mathcal{F}_V$ is comeagre in $\mathcal{F}_V$ for all $f \in \mathcal{F}_U$.

Let us spell out some of the consequences of Theorem 1.1. For instance, let $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be any non-identity element such that $\mathrm{fix}(\tilde{f}) = \emptyset$ and let $V = [0, 1]$, say. Then there are many automorphisms $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$, in the sense of category, such that $\mathrm{fix}(\tilde{g}) = V$ and $\{f, g\}$ topologically generates $\mathrm{Aut}(\mathbb{Q}, \leqslant)$.

Theorem 1.1 also has several limitations. If U and V are closed subsets of $\mathbb{R}$ with non-empty intersection, then the subgroup $\langle f, g \rangle$ is not dense for any $f \in \mathcal{F}_U$ and $g \in \mathcal{F}_V$ (as the extension of every product of f and g to a homeomorphism on $\mathbb{R}$ has fixed points in $U \cap V$). In Example 3.4, we give some further conditions on disjoint closed subsets U and V of $\mathbb{R}$ such that $\mathcal{D}_f \cap \mathcal{F}_V$ is not comeagre in $\mathcal{F}_V$ for some $f \in \mathcal{F}_U$. Note that there exist disjoint closed subsets U and V of $\mathbb{R}$ that do not satisfy the hypothesis of Theorem 1.1. For example, if $U = (-\infty, -1] \cup [1, \infty)$ and V is any closed subset of $\mathbb{R}$ disjoint from U, then the unique component $(-1, 1)$ of $\mathbb{R} \setminus U$ is bounded but $\mathbb{R} \setminus V$ must contain a component $[\alpha, \infty)$ for some $\alpha \in \mathbb{R} \cup \{-\infty\}$ with $\alpha < 1$.

A simple application of the Kuratowski-Ulam Theorem [8, Theorem 8.41] to Theorem 1.1 gives us the following corollary.

Corollary 1.2. *Let U and V be disjoint closed subsets of $\mathbb{R}$ satisfying* (i) *or* (ii) *from Theorem 1.1. Then the set of pairs in $\mathcal{F}_U \times \mathcal{F}_V$ that topologically generate $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ is comeagre in $\mathcal{F}_U \times \mathcal{F}_V$.*

Macpherson [12] proved that any oligomorphic closed subgroup of S_∞ contains a free subgroup of infinite rank; see also Gartside and Knight [5] and Kechris [9]. In particular, Macpherson's result applies in the case of $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ and $\operatorname{Aut}(R_{\mathcal{C}})$. We deduce a similar result from Corollary 1.2 as follows. As in Corollaries 3.6 and 3.7 from [3], it can be shown that there exists a compact set $K \subseteq \mathcal{F}_\emptyset$ such that $|K| = 2^{\aleph_0}$, K freely generates a free subgroup of $\operatorname{Aut}(\mathbb{Q}, \leqslant)$, and there exist $f, g \in K$ that topologically generate $\operatorname{Aut}(\mathbb{Q}, \leqslant)$. Consequently, for all κ with $2 \leqslant \kappa \leqslant 2^{\aleph_0}$ there exists a dense free subgroup of $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ with rank κ.

In the next theorem we show that $\mathcal{D}_f \neq \emptyset$ for all non-identity $f \in \operatorname{Aut}(\mathbb{Q}, \leqslant)$.

Theorem 1.3. *Let $f \in \operatorname{Aut}(\mathbb{Q}, \leqslant)$ be a non-identity element. Then there exists $g \in \operatorname{Aut}(\mathbb{Q}, \leqslant)$ such that f and g topologically generate $\operatorname{Aut}(\mathbb{Q}, \leqslant)$.*

Various global results can be obtained as corollaries of Theorem 1.1, such as the following that we will prove in Section 3.

Corollary 1.4. *Let $f \in \operatorname{Aut}(\mathbb{Q}, \leqslant)$ be a non-identity element such that $\mathbb{R} \setminus \operatorname{fix}(\tilde{f})$ is unbounded above and below and let $\mathcal{F}$ be the set of all $g \in \operatorname{Aut}(\mathbb{Q}, \leqslant)$ such that $\operatorname{fix}(\tilde{f}) \cap \operatorname{fix}(\tilde{g}) = \emptyset$. Then $\mathcal{D}_f \cap \mathcal{F}$ is comeagre in $\mathcal{F}$.*

Following the comments in the introduction, the set

$$\{\, (f, g) \in \operatorname{Aut}(\mathbb{Q}, \leqslant) \times \operatorname{Aut}(\mathbb{Q}, \leqslant) \;:\; f, g \text{ topologically generate } \operatorname{Aut}(\mathbb{Q}, \leqslant) \,\}$$

is nowhere dense in $\operatorname{Aut}(\mathbb{Q}, \leqslant) \times \operatorname{Aut}(\mathbb{Q}, \leqslant)$. The following theorem indicates that something stronger is true: two elements of $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ chosen at random generate a nowhere dense subgroup of $\operatorname{Aut}(\mathbb{Q}, \leqslant)$. This result contrasts with Corollaries 1.2 and 1.4.

Theorem 1.5. *The set of pairs in $\operatorname{Aut}(\mathbb{Q}, \leqslant) \times \operatorname{Aut}(\mathbb{Q}, \leqslant)$ that generate a nowhere dense subgroup of $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ is comeagre in $\operatorname{Aut}(\mathbb{Q}, \leqslant) \times \operatorname{Aut}(\mathbb{Q}, \leqslant)$.*

1.2. **The random graph.** The *orbit-type of $f \in \operatorname{Aut}(R_{\mathcal{C}})$* is the sequence $(k_\infty, k_1, k_2, \ldots)$ where f has exactly k_i distinct orbits of length i for all $1 \leq i \leq \infty$. Let

$$\mathcal{I} = \{\, f \in \operatorname{Aut}(R_{\mathcal{C}}) \;:\; f \text{ has orbit-type } (k, 0, 0, \ldots) \text{ for some } k \in \mathbb{N} \cup \{\infty\} \,\}.$$

If Σ is a finite set of vertices in $R_{\mathcal{C}}$, then define

$$\mathcal{I}_\Sigma = \{\, f \in \mathcal{I} \;:\; \Sigma \text{ is a set of representatives of all the orbits of } f \,\}.$$

It can be readily verified that $\mathcal{I}$ is closed in $\operatorname{Aut}(R_{\mathcal{C}})$ and each $\mathcal{I}_\Sigma$ is G_δ in $\operatorname{Aut}(R_{\mathcal{C}})$. Hence, $\mathcal{I}$ and $\mathcal{I}_\Sigma$ are Polish spaces. Further details about the $\mathcal{C}$-coloured random graph $R_{\mathcal{C}}$ and its automorphisms can be found in Section 2.

The following theorem is our main result concerning the $\mathcal{C}$-coloured random graph $R_{\mathcal{C}}$.

Theorem 1.6. *Let $f \in \operatorname{Aut}(R_{\mathcal{C}})$ be a non-identity element and let Σ be a finite set of vertices in $R_{\mathcal{C}}$. Then $\mathcal{D}_f \cap \mathcal{I}_\Sigma$ is comeagre in $\mathcal{I}_\Sigma$.*

Let us consider some of the consequences of Theorem 1.6. If $f \in \mathrm{Aut}(R_{\mathcal{C}})$ is any non-identity element and $k \in \mathbb{N}$ is arbitrary, then, by Theorem 1.6, there exists $g \in \mathcal{I}$ with exactly k infinite orbits such that f and g topologically generate $\mathrm{Aut}(R_{\mathcal{C}})$. By a small modification of the proof of Theorem 1.6 we can also prove the following corollary where $\mathcal{I}_\infty = \{ f \in \mathrm{Aut}(R_{\mathcal{C}}) : f \text{ has orbit-type } (\infty, 0, 0, \ldots) \}$.

Corollary 1.7. *Let $f \in \mathrm{Aut}(R_{\mathcal{C}})$ be a non-identity element. Then $\mathcal{D}_f \cap \mathcal{I}$ is comeagre in $\mathcal{I}$. Moreover, $\mathcal{I}_\infty$ is comeagre in $\mathcal{I}$ and so $\mathcal{D}_f \cap \mathcal{I}_\infty$ is comeagre in $\mathcal{I}$.*

The following corollary is a consequence of Corollary 1.7 and the remarks preceding it.

Corollary 1.8. *Let $f \in \mathrm{Aut}(R_{\mathcal{C}})$ be any non-identity element and let $k \in \mathbb{N} \cup \{\infty\}$ be arbitrary. Then there exists $g \in \mathrm{Aut}(R_{\mathcal{C}})$ with orbit-type $(k, 0, 0, \ldots)$ such that f and g topologically generate $\mathrm{Aut}(R_{\mathcal{C}})$.*

As in Corollary 1.2, by applying the Kuratowski-Ulam Theorem to Corollary 1.7, we obtain the following.

Corollary 1.9. *The set of pairs in $\mathcal{I} \times \mathcal{I}$ that topologically generate $\mathrm{Aut}(R_{\mathcal{C}})$ is comeagre in $\mathcal{I} \times \mathcal{I}$.*

As in the comments after Corollary 1.2, using analogous arguments to those given in [3, Corollaries 3.6 and 3.7], Corollary 1.9 can be used to prove that there exists a compact set $K \subseteq \mathcal{I}$ such that $|K| = 2^{\aleph_0}$, K freely generates a free subgroup of $\mathrm{Aut}(R_{\mathcal{C}})$, and there exist $f, g \in K$ that topologically generate $\mathrm{Aut}(R_{\mathcal{C}})$. Consequently, for all κ with $2 \leqslant \kappa \leqslant 2^{\aleph_0}$ there exists a dense free subgroup of $\mathrm{Aut}(R_{\mathcal{C}})$ with rank κ.

As was the case with $\mathrm{Aut}(\mathbb{Q}, \leqslant)$, two elements of $\mathrm{Aut}(R_{\mathcal{C}})$ chosen at random generate a nowhere dense subgroup of $\mathrm{Aut}(R_{\mathcal{C}})$.

Theorem 1.10. *The set of pairs in $\mathrm{Aut}(R_{\mathcal{C}}) \times \mathrm{Aut}(R_{\mathcal{C}})$ that generate a nowhere dense subgroup of $\mathrm{Aut}(R_{\mathcal{C}})$ is comeagre in $\mathrm{Aut}(R_{\mathcal{C}}) \times \mathrm{Aut}(R_{\mathcal{C}})$.*

We conclude this section with some open questions. Is it possible to find necessary and sufficient conditions on disjoint closed subsets of $\mathbb{R}$ such that $\mathcal{D}_f \cap \mathcal{F}_V$ is comeagre in $\mathcal{F}_V$ for all $f \in \mathcal{F}_U$? In [3, Theorem 3.3(iii)] we proved that if $f \in S_\infty$ has no finite cycles, then $\{ g \in S_\infty : f, g \text{ topologically generate } S_\infty \}$ is comeagre in S_∞. We ask: is it true that if $f \in \mathcal{I}$, then $\mathcal{D}_f$ is comeagre in $\mathrm{Aut}(R_{\mathcal{C}})$? It is also natural to ask if analogous results can be obtained for arbitrary automorphism groups of Fraïssé limits. In particular, if K is a Fraïssé limit, then is it true that $\mathrm{Aut}(K)$ is always topologically 2-generated? If $f \in \mathrm{Aut}(K)$ is arbitrary, then does there exist $g \in \mathrm{Aut}(K)$ such that f and g topologically generate $\mathrm{Aut}(K)$?

2. Preliminaries

In this section we introduce the background material and notation required to prove our main theorems.

A topological group G is called a *Polish group* if the topology on G is complete and separable. If G is a Polish group and A is a subset of G, then we will say that *A topologically generates G* if the group $\langle A \rangle$ generated by A is dense in G.

Let X be any set and let $\mathrm{Sym}(X)$ denote the symmetric group on X. If X is countably infinite, then we may write S_∞ instead of $\mathrm{Sym}(X)$. We refer the reader to [8] for basic properties of S_∞ and Polish groups in general.

If $f : X \longrightarrow X$ is any function, and $Y \subseteq X$, then denote by $f|_Y$ the *restriction* of f to Y. If $Y \subseteq Y'$, then we will say that $f|_{Y'}$ is an *extension* of $f|_Y$. If A is a subset of $\mathrm{Sym}(X)$, then define

$$A^{<\infty} = \{\, f|_Y \ :\ f \in A \text{ and } Y \text{ is a finite subset of } X \,\}.$$

If $f \in A^{<\infty}$, then we will denote the *domain* of f by $\mathrm{dom}(f)$ and the range of f by $\mathrm{ran}(f)$.

If X is any set and $f \in \mathrm{Sym}(X)$, then the *fix* of f is the set

$$\mathrm{fix}(f) = \{\, \alpha \in X \ :\ f(\alpha) = \alpha \,\},$$

and the *support* of f is

$$\mathrm{supp}(f) = \{\, \alpha \in X \ :\ f(\alpha) \neq \alpha \,\} = X \setminus \mathrm{fix}(f).$$

If $\alpha \in X$ and $f \in \mathrm{Sym}(X)$, then the *orbit of* α *under* f is simply the set

$$\mathrm{Orb}(f, \alpha) = \{\, f^i(\alpha) \ :\ i \in \mathbb{Z} \,\}.$$

If $f \in \mathrm{Sym}(X)^{<\infty}$ and $\alpha \in \mathrm{dom}(f) \cup \mathrm{ran}(f)$, then denote by

$$\mathrm{Orb}(f, \alpha) = \{\, f^i(\alpha) \ :\ i \in \mathbb{Z} \text{ where } f^i(\alpha) \text{ is defined} \,\}.$$

If $f \in \mathrm{Sym}(X)^{<\infty}$, then we will refer to $\mathrm{Orb}(f, \alpha)$ as the *partial orbit* of α under f. If $f \in \mathrm{Sym}(X)^{<\infty}$ or $f \in \mathrm{Sym}(X)$, then $\mathrm{Orb}(f, \alpha)$ is called a *cycle* if there exists $n \neq 0$ such that $f^n(\alpha) = \alpha$. The *length* of a cycle is the least such $n \in \mathbb{N}$. If $\alpha \notin \mathrm{dom}(f) \cup \mathrm{ran}(f)$, then we define $\mathrm{Orb}(f, \alpha) = \emptyset$. The *orbit-type of* f is the sequence $(k_\infty, k_1, k_2, \ldots)$ where f has exactly k_i distinct orbits of length i for all $1 \leq i \leq \infty$.

All the groups considered here are subgroups of S_∞. The group S_∞ endowed with the topology of pointwise convergence and the discrete topology on the underlying set is a Polish group. If $f \in S_\infty^{<\infty}$, then we define

$$[f] = \{\, g \in S_\infty \ :\ g|_{\mathrm{dom}(f)} = f \,\}.$$

Note that the sets $[f]$ form a basis for the topology on S_∞.

A function $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ is an *order-preserving automorphism* of $\mathbb{Q}$ if it is a bijection and $f(\alpha) < f(\beta)$ whenever $\alpha, \beta \in \mathbb{Q}$ and $\alpha < \beta$. The group of automorphisms of $\mathbb{Q}$ (with operation the composition of functions) is denoted by $\mathrm{Aut}(\mathbb{Q}, \leqslant)$. Since $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ is a closed subgroup of $\mathrm{Sym}(\mathbb{Q})$ with the topology of pointwise convergence and $\mathbb{Q}$ endowed with the discrete topology, it follows that $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ is a Polish group too.

If $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$, then there exists a unique order-preserving homeomorphism $\tilde{f}$ of $\mathbb{R}$ extending f. Throughout the paper, whenever we refer to a subset of $\mathbb{R}$ as closed, open, G_δ, and so on, the topology in question is the usual Euclidean topology on $\mathbb{R}$. If $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$, then $\mathrm{fix}(\tilde{f})$ is a closed subset of $\mathbb{R}$. Conversely, if U is a closed subset of $\mathbb{R}$, then there exists $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ such that $U = \mathrm{fix}(\tilde{f})$.

Any non-identity automorphism of $\mathbb{Q}$ has orbit-type

$$(\infty, k_1, 0, 0, \ldots)$$

for some $k_1 \in \{0,1,2,\ldots\} \cup \{\infty\}$. Due to their limited structure, orbit-types are not a particularly useful tool when considering $\mathrm{Aut}(\mathbb{Q}, \leqslant)$. Hence we require the following alternative notion introduced by Truss [17]. We modify Truss' original definition slightly to better suit our present work.

If U is a closed subset of $\mathbb{R}$, then a maximal open interval in $\mathbb{R}\setminus U$ will be referred to as a *component* of $\mathbb{R} \setminus U$. Let $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$. Then an *orbital of* f is a component of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ or a singleton set consisting of an element in $\mathrm{fix}(\tilde{f})$. We will freely use the following facts about orbitals without reference.

Proposition 2.1. *Let Σ be an orbital of $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ and let $\alpha \in \Sigma \cap \mathbb{Q}$. Then the following hold:*

(i) $\Sigma \cap \mathbb{Q} = \{\, \beta \in \mathbb{Q} \,:\, f^m(\alpha) \leqslant \beta \leqslant f^n(\alpha)$ *for some* $m, n \in \mathbb{Z} \,\}$;
(ii) $\sup\{\, f^n(\alpha) \,:\, n \in \mathbb{Z} \,\} = \sup(\Sigma)$ *and* $\inf\{\, f^n(\alpha) \,:\, n \in \mathbb{Z} \,\} = \inf(\Sigma)$;
(iii) *if* $\sup(\Sigma) < \infty$, *then* $\sup(\Sigma) \in \mathrm{fix}(\tilde{f})$, *if* $\inf(\Sigma) > -\infty$, *then* $\inf(\Sigma) \in \mathrm{fix}(\tilde{f})$.

Obviously the orbitals of $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ partition $\mathbb{R}$ and if Σ is an orbital of f, then $\tilde{f}(\alpha) \in \Sigma$ for all $\alpha \in \Sigma$. The intersection $\Sigma \cap \mathbb{Q}$ from the previous proposition is the original definition of the orbital of α under f given by Truss.

The *parity* of an orbital Σ is 0 if Σ is a singleton set, -1 if $\tilde{f}(\alpha) < \alpha$ for some (and hence all) $\alpha \in \Sigma$ and $+1$ if $\tilde{f}(\alpha) > \alpha$ for some $\alpha \in \Sigma$. One can readily verify that the parity of an orbital is a well-defined notion. The notions of orbitals and their parity can be used to describe a generic order-automorphism of $\mathbb{Q}$.

Proposition 2.2. *Let $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be a generic automorphism. Then the sets of orbitals of f of parity $+1$, -1, and 0 are each densely linearly ordered without endpoints, and each is dense in the union of the other two.*

For a proof see [17, Theorem 4.1].

The rationals $\mathbb{Q}$ are a *ultrahomogeneous* structure, that is, any isomorphism of finite substructures of $\mathbb{Q}$ can be extended to an element of $\mathrm{Aut}(\mathbb{Q}, \leqslant)$. If $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$ and $\alpha \in \mathrm{dom}(f)$, then the *parity of* f on $\mathrm{Orb}(f, \alpha)$ is -1 if $f(\alpha) < \alpha$, $+1$ if $f(\alpha) > \alpha$, and 0 if $f(\alpha) = \alpha$.

If U is a closed subset of $\mathbb{R}$, then define

$$\mathcal{F}_U = \{\, f \in \mathrm{Aut}(\mathbb{Q}, \leqslant) \,:\, \mathrm{fix}(\tilde{f}) = U \,\}.$$

If $f \in \mathcal{F}_U^{<\infty}$, then the parities of any two partial orbits in a given component of $\mathbb{R} \setminus U$, if they exist, are equal.

The *random graph* is the unique, up to isomorphism, countably infinite graph such that if U and V are any disjoint finite sets of vertices, then there exists a vertex adjacent to every element of U and to no element in V. The *$\mathcal{C}$-coloured random graph* $R_{\mathcal{C}}$ is a generalization of the random graph where edges and non-edges are replaced by edges coloured by the set $\mathcal{C}$. To be more precise, we let Ω be a countably infinite set, $\mathcal{C} = \{\lambda_1, \ldots\}$ be a finite set, and F be any function from the two-element subsets of Ω into $\mathcal{C}$. Then we say that $R_{\mathcal{C}} = (\Omega, \mathcal{C}, F)$ is the *$\mathcal{C}$-coloured random graph* if the following property is satisfied:

> *if* $(U_1, \ldots, U_t)$ *is any tuple of disjoint finite subsets of* Ω, *then there exists* $\beta \in \Omega$ *such that* $F\{\alpha, \beta\} = \lambda_i$ *for all* $\alpha \in U_i$ *and for all* $1 \leq i \leq |\mathcal{C}|$.

We refer to the above as the *Alice's restaurant property* (i.e. you can have whatever you want at Alice's restaurant), the element β as a *witness*, and the set Ω as the *vertices* of $R_{\mathcal{C}}$.

A function $f : R_{\mathcal{C}} \longrightarrow R_{\mathcal{C}}$ is an *automorphism* of $R_{\mathcal{C}}$ if f is a bijection and $F\{f(\alpha), f(\beta)\} = F\{\alpha, \beta\}$ for all $\alpha, \beta \in \Omega$. We will denote the group of automorphisms of $R_{\mathcal{C}}$ by $\mathrm{Aut}(R_{\mathcal{C}})$.

It is straightforward to prove that if $\mathcal{C}$ is fixed, then any two $\mathcal{C}$-coloured random graphs are isomorphic. Hence we can meaningfully talk about *the* $\mathcal{C}$-coloured random graph and for any given $\mathcal{C}$ we will work with a fixed copy of $R_{\mathcal{C}}$. Of course, if $|\mathcal{C}| = 1$, then $\mathrm{Aut}(R_{\mathcal{C}})$ is just the symmetric group $\mathrm{Sym}(\Omega)$ on Ω. As mentioned in the introduction, a detailed analysis of pairs of elements in $\mathrm{Sym}(\Omega)$ that topologically generating $\mathrm{Sym}(\Omega)$ can be found in [3]. Hence in the following we will always assume that $|\mathcal{C}| \geqslant 2$.

As above, $\mathrm{Aut}(R_{\mathcal{C}})$ is a closed subgroup of $\mathrm{Sym}(\Omega)$, the symmetric group on Ω endowed with the pointwise convergence topology. Hence $\mathrm{Aut}(R_{\mathcal{C}})$ is a Polish group.

Although any sequence $(k_\infty, k_1, \ldots)$ with $k_1 \cdot 1 + k_2 \cdot 2 + \cdots = \infty$ describes the orbit-type of some $f \in S_\infty$, the sequences describing the orbit-types of elements of $\mathrm{Aut}(R_{\mathcal{C}})$ are more complicated. For example, $(1, 1, 0, 0, \ldots)$ is not the orbit-type of any $f \in \mathrm{Aut}(R_{\mathcal{C}})$ but there are $2^{\aleph_0}$ non-conjugate elements with orbit-type $(1, 0, 0, \ldots)$ [16]. Truss gave a complete characterization of possible orbit-types of elements of $\mathrm{Aut}(R_{\mathcal{C}})$ and showed that $\mathrm{Aut}(R_{\mathcal{C}})$ is simple in [16].

Similarly to the rationals, any isomorphism of finite substructures of $R_{\mathcal{C}}$ can be extended to an element of $\mathrm{Aut}(R_{\mathcal{C}})$, that is, $R_{\mathcal{C}}$ is ultrahomogeneous.

Let

$$\mathcal{I} = \{\, f \in \mathrm{Aut}(R_{\mathcal{C}}) \,:\, f \text{ has orbit-type } (k, 0, 0, \ldots) \text{ for some } k \in \mathbb{N} \cup \{\infty\} \,\}.$$

Then for all $f \in \mathcal{I}^{<\infty}$ no partial orbit of f is a cycle.

In the next proposition, we give two facts about elements of $\mathrm{Aut}(R_{\mathcal{C}})$ and $\mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$ that we require later. The proof is left to the reader.

Proposition 2.3. *Let $f \in \mathrm{Aut}(R_{\mathcal{C}})$ be any non-identity element, let $g \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$, and let Σ be a finite subset of the vertices Ω of $R_{\mathcal{C}}$. Then*

(i) *if g has n partial orbits, then there exist $\alpha, \beta \in \Omega \setminus [\Sigma \cup \mathrm{dom}(g) \cup \mathrm{ran}(g) \cup \mathrm{fix}(f)]$ such that the extension h of g defined by $h(\alpha) = \beta$ has $n + 1$ partial orbits and belongs to $\mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$;*

(ii) *if $\alpha \in \mathrm{ran}(g) \setminus \mathrm{dom}(g)$, then there exists $\beta \in \Omega \setminus [\Sigma \cup \mathrm{fix}(f)]$ such that the extension h of g defined by $h(\alpha) = \beta$ belongs to $\mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$.*

We will make repeated use of the following routine lemma.

Lemma 2.4. *Let X be an infinite set, let G be a Polish subgroup of $\mathrm{Sym}(X)$, let $f, g \in G$, and let $h \in G^{<\infty}$ be such that $\langle\, f, g \,\rangle \cap [h] \neq \emptyset$. Then there exists a finite subset Y of X such that $\langle\, f, k \,\rangle \cap [h] \neq \emptyset$ for all $k \in [g|_Y]$.* □

3. THE RATIONALS - THE PROOFS

We prove Theorems 1.3, its corollaries, and Theorem 1.1 in a series of lemmas.

Lemma 3.1. *Let $f, g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be non-identity elements such that $\mathrm{fix}(\tilde{f}) \cap \mathrm{fix}(\tilde{g}) = \emptyset$ and let $\alpha, \beta \in \mathbb{Q}$. Then there exist $h, k \in \langle\, f, g \,\rangle$ such that $h(\alpha) \geqslant \beta$ and $k(\alpha) \leqslant \beta$.*

Proof. We will prove that there exists $h \in \langle f, g\rangle$ such that $h(\alpha) \geqslant \beta$. Hence we may assume that $\alpha < \beta$. As $\mathrm{fix}(\tilde{f}) \cap \mathrm{fix}(\tilde{g}) = \emptyset$, either α belongs to an infinite orbital of f or an infinite orbital of g. Without loss of generality assume the former. Let $\alpha_0 = \alpha$ and let (σ_0, τ_0) in $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ be the orbital of f containing α_0. If $\tau_0 > \beta$, then there exists $m \in \mathbb{Z}$ such that $f^m(\alpha) > \beta$ and the proof is complete.

If $\tau_0 \leqslant \beta$, then as $\tau_0 \in \mathrm{fix}(\tilde{f})$ there exists an orbital (σ_1, τ_1) in $\mathbb{R} \setminus \mathrm{fix}(\tilde{g})$ of g such that $\tau_0 \in (\sigma_1, \tau_1)$. Choose $n_0 \in \mathbb{Z}$ such that $\alpha_1 = f^{n_0}(\alpha_0) \in (\sigma_1, \tau_0)$. If $\tau_1 > \beta$, then there exists $m \in \mathbb{Z}$ such that $g^m(\alpha_1) > \beta$ and the proof is complete.

If $\tau_1 \leqslant \beta$, then there exists a component (σ_2, τ_2) of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ such that $\tau_1 \in (\sigma_2, \tau_2)$. Choose $n_1 \in \mathbb{Z}$ such that $\alpha_2 = g^{n_1}(\alpha_1) \in (\sigma_2, \tau_1)$. Note that as $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ and $\tau_0 < \tau_1 < \tau_2$, the orbitals (σ_0, τ_0) and (σ_2, τ_2) are disjoint and so $\alpha_1 < \tau_0 < \sigma_2 < \alpha_2$.

We continue this process by choosing components $(\sigma_{n+1}, \tau_{n+1})$ alternately in $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ and $\mathbb{R} \setminus \mathrm{fix}(\tilde{g})$ such that $\tau_n \in (\sigma_{n+1}, \tau_{n+1})$ and $\alpha_{n+1} \in (\sigma_{n+1}, \tau_n)$. If $\tau_n > \beta$ at some step, then the proof is completed as above. Otherwise, by construction $\tau_1 < \tau_2 < \cdots$ and so the orbitals (σ_n, τ_n) and $(\sigma_{n+2}, \tau_{n+2})$ are disjoint. Hence $\alpha_n < \tau_{n-1} < \sigma_{n+1} < \alpha_{n+1}$ where $\tau_{n-1}, \sigma_{n+1} \in \mathrm{fix}(\tilde{f})$ if n is odd and $\tau_{n-1}, \sigma_{n+1} \in \mathrm{fix}(\tilde{g})$ if n is even. Thus the sequence $\alpha_1, \alpha_2, \ldots$ is strictly increasing and bounded above by β. Hence $\alpha_1, \alpha_2, \ldots$ converges and its limit belongs to $\mathrm{fix}(\tilde{g})$ and $\mathrm{fix}(\tilde{f})$, a contradiction. □

Lemma 3.2. *Let $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be any non-identity element with $\mathrm{fix}(\tilde{f}) \neq \emptyset$, let V be a non-empty closed subset of $\mathbb{R}$ disjoint from $\mathrm{fix}(\tilde{f})$, let $g \in \mathcal{F}_V^{<\infty}$, and let Σ be a finite subset of $\mathbb{Q}$. Then there are $h \in \mathcal{F}_V \cap [g]$ and $k \in \langle\, f, h \,\rangle$ such that $k(\Sigma)$ is contained in a single component of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$.*

Proof. If $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ contains an unbounded component and $h \in \mathcal{F}_V \cap [g]$ is arbitrary, then, by Lemma 3.1, there exists $k \in \langle\, f, h \,\rangle$ such that $k(\Sigma)$ is contained in that unbounded component.

If $\mathbb{R} \setminus V$ contains an unbounded component $(-\infty, \alpha)$ or (α, ∞) and $h \in \mathcal{F}_V \cap [g]$ is arbitrary, then, again by Lemma 3.1, there exists $k \in \langle\, f, h \,\rangle$ such that $k(\Sigma)$ is contained in $(-\infty, \alpha)$ or (α, ∞). Since V is non-empty, it follows that $\alpha \in \mathbb{R}$ and so there exists $n \in \mathbb{Z}$ such that $h^n k(\Sigma)$ is contained in the component of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ containing α.

It remains to prove the lemma when all the components in $\mathbb{R} \setminus V$ and $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ are bounded. Since V and $\mathrm{fix}(\tilde{f})$ are disjoint, $\mathbb{R} \setminus V$ and $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ are unbounded above and below. Assume without loss of generality that Σ is contained in $\mathrm{dom}(g)$ and let $\Sigma' = \mathrm{dom}(g) \cup \mathrm{ran}(g)$. Since every component of $\mathbb{R} \setminus V$ is bounded, there exists $\alpha \in \mathbb{Q} \setminus V$ with $\max \Sigma' < \alpha$ and where α and $\max \Sigma'$ are not in the same component of $\mathbb{R} \setminus V$. By Lemma 3.1, if $h_0 \in \mathcal{F}_V \cap [g]$ is arbitrary, then there exists $k_0 \in \langle\, f, h_0 \,\rangle$ such that $k_0(\min(\Sigma')) \geqslant \alpha$. In particular, we may fix $h_0 \in \mathcal{F}_V \cap [g]$ so that it has parity $+1$ on the components of $\mathbb{R} \setminus V$ not containing elements in Σ'. We may also fix $m_0, m_1, \ldots, m_{2i+1} \in \mathbb{Z}$ such that if $k_0 = f^{m_{2i+1}} h_0^{m_{2i}} \cdots f^{m_1} h_0^{m_0}$, then $k_0(\min(\Sigma')) \geqslant \alpha$.

By Lemma 2.4, the value of k_0 on all the elements of Σ' is determined by the values of f and h_0 on a finite subset of $\mathbb{Q}$ containing Σ'. Let Γ denote the union of this finite set and $k_0(\Sigma')$. Then there exists $\beta \in \mathbb{Q}$ such that $\beta > \max\Gamma$ and β lies in a component (σ_1, σ_2) of $\mathbb{R} \setminus V$ with $\max\Gamma \notin (\sigma_1, \sigma_2)$. Note that since $k_0(\min(\Sigma')) \geqslant \alpha$ and $k_0(\min(\Sigma')) \in \Gamma$, it follows that $\beta > \alpha$. Let (τ_1, τ_2) be the component of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ containing σ_2 and let (σ_3, σ_4) be the component of $\mathbb{R} \setminus V$ containing τ_2. Note that $\sigma_4 > \tau_2 > \sigma_2$ and so $(\sigma_1, \sigma_2) \cap (\sigma_3, \sigma_4) = \emptyset$. Let $h \in \mathcal{F}_V \cap [g]$ be equal to h_0 except on (σ_3, σ_4) where h has parity -1. Then $h_0|_\Gamma = h|_\Gamma$ and so $k_0|_{\Sigma'} = (f^{m_{2i+1}} h^{m_{2i}} \cdots f^{m_1} h^{m_0})|_{\Sigma'}$.

Note that if $n_1, n_3 \ldots, n_{2j+1} \in \mathbb{Z}$ and $n_0, n_2, n_4, \ldots, n_{2j} \geqslant 0$, then

$$f^{n_{2j+1}} h^{n_{2j}} \cdots f^{n_1} h^{n_0}(\beta) \leqslant \tau_2 \tag{1}$$

by the choices of the parities of orbitals of h. Also note that h has parity $+1$ on any component (δ, γ) of $\mathbb{R} \setminus V$ where δ is at least the infimum of the component of $\mathbb{R} \setminus V$ containing α, and $\gamma \leqslant \sigma_3$ from the definitions of h and h_0.

We will find $k_1 \in \langle f, h \rangle$ such that $k_1(\alpha)$ and $k_1(\beta)$ are in the same component of $\mathbb{R} \setminus V$. Recall, from their definitions, that α and β are in different components of $\mathbb{R}\setminus V$ and $\beta > \alpha$. Let $\alpha_0 = \alpha$ and let (μ_0, ν_0) be the component of $\mathbb{R}\setminus V$ containing α_0 for some $\mu_0, \nu_0 \in \mathbb{R}$. By assumption, there exists a component (μ_1, ν_1) of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ such that $\nu_0 \in (\mu_1, \nu_1)$ as $\nu_0 \in \mathrm{fix}(\tilde{h}) = V$. Since the parity of h on (μ_0, ν_0) is 1, there exists $n_0 > 0$ such that $\alpha_1 = h^{n_0}(\alpha_0) \in (\mu_1, \nu_0)$ and $\alpha_1 > \alpha_0$.

Let (μ_2, ν_2) be the component of $\mathbb{R} \setminus V$ such that $\nu_1 \in (\mu_2, \nu_2)$ and let $n_1 \in \mathbb{Z}$ be such that $\alpha_2 = f^{n_1}(\alpha_1) \in (\mu_2, \nu_1)$ and $\alpha_2 > \alpha_1$. If $f^{n_1} h^{n_0}(\alpha)$ and $f^{n_1} h^{n_0}(\beta)$ are in the same component of $\mathbb{R} \setminus V$, then we set $k_1 = f^{n_1} h^{n_0}$. Otherwise, there exists $n_2 > 0$ such that $\alpha_3 = h^{n_2}(\alpha_2) \in (\mu_3, \nu_2)$ where (μ_3, ν_3) is the component of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ with $\nu_2 \in (\mu_3, \nu_3)$.

We repeat this process. Assume that

$$f^{n_{2j+1}} h^{n_{2j}} \cdots f^{n_1} h^{n_0}(\alpha) \text{ and } f^{n_{2j+1}} h^{n_{2j}} \cdots f^{n_1} h^{n_0}(\beta)$$

are not in the same component of $\mathbb{R} \setminus V$ for all j. Then the sequence $\alpha_0, \alpha_1, \ldots$ is strictly increasing and bounded above by τ_2 from (1). If $t > 0$ is even, then $\alpha_t < \mu_{t+1} < \alpha_{t+1}$ and $\mu_{t+1} \in \mathrm{fix}(\tilde{f})$. If t is odd, then $\alpha_t < \nu_{t-1} < \alpha_{t+1}$ and $\nu_{t-1} \in V$. Hence the limit of the sequence $\alpha_0, \alpha_1, \ldots$ belongs to $\mathrm{fix}(\tilde{f})$ and V, a contradiction.

It follows that $f^{n_{2j+1}} h^{n_{2j}} \cdots f^{n_1} h^{n_0}(\alpha)$ and $f^{n_{2j+1}} h^{n_{2j}} \cdots f^{n_1} h^{n_0}(\beta)$ are in the same component of $\mathbb{R} \setminus V$ for some j and we set $k_1 = f^{n_{2j+1}} h^{n_{2j}} \cdots f^{n_1} h^{n_0}$.

As $k_0(\min\Sigma') \geqslant \alpha$ and $k_0(\max\Sigma') \leqslant \max\Gamma < \beta$, we have that $k_1 k_0(\Sigma')$ is contained in a single component (μ, ν) of $\mathbb{R} \setminus V$. Finally, as all the components of $\mathbb{R}\setminus V$ and $\mathbb{R}\setminus\mathrm{fix}(\tilde{f})$ are bounded, there is a component (δ, γ) of $\mathbb{R}\setminus\mathrm{fix}(\tilde{f})$ containing μ. Thus there exists $r \in \mathbb{Z}$ such that $h^r k_1 k_0(\Sigma') \subseteq (\mu, \gamma)$. Therefore if $k = h^r k_1 k_0$, then $k(\Sigma) \subseteq k(\Sigma')$ is contained in the single component (δ, γ) of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$. □

Let $\mathcal{P}$ denote the set of all $p \in \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$ such that $\max\mathrm{dom}(p) < \min\mathrm{ran}(p)$. Note that any extension of p has parity $+1$ on all of the partial orbits containing elements of $\mathrm{dom}(p) \cup \mathrm{ran}(p)$. Let G be any subgroup of $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ and let $p \in \mathcal{P}$. Then $G \cap [p^{-1}] \neq \emptyset$ if and only if $G \cap [p] \neq \emptyset$. It is easy to see that if $G \cap [p] \neq \emptyset$ for all $p \in \mathcal{P}$, then G is dense in $\mathrm{Aut}(\mathbb{Q}, \leqslant)$.

Proof of Theorem 1.1. Let $f \in \mathcal{F}_U$ be arbitrary. It suffices to prove that

$$A_p = \{\, g \in \mathcal{F}_V \;:\; \langle\, f, g \,\rangle \cap [p] \neq \emptyset \,\}$$

is open and dense in $\mathcal{F}_V$ for all $p \in \mathcal{P}$, as

$$\mathcal{D}_f \cap \mathcal{F}_V = \bigcap_{p \in \mathcal{P}} A_p.$$

In any case, by Lemma 2.4, A_p is open for all $p \in \mathcal{P}$. Hence it remains to show that A_p is dense in $\mathcal{F}_V$. To this end, fix $p \in \mathcal{P}$, let $g_0 \in \mathcal{F}_V^{<\infty}$ be arbitrary, and let $\Sigma_0 = \operatorname{dom}(p) \cup \operatorname{ran}(p)$.

(i). *all the components of* $\mathbb{R} \setminus U$ *and* $\mathbb{R} \setminus V$ *are bounded.* By Lemmas 2.4 and 3.2, there exists an extension $h_0 \in \mathcal{F}_V^{<\infty}$ of g_0 such that for some $m_1, m_2, \ldots, m_j \in \mathbb{Z}$ the set

$$\Sigma_1 = h_0^{m_j} f^{m_{j-1}} \cdots h_0^{m_2} f^{m_1}(\Sigma_0)$$

is contained in a single component (α_1, β_1) of $\mathbb{R} \setminus \operatorname{fix}(\tilde{f}) = \mathbb{R} \setminus U$. Then, from the assumption that (i) holds, there exist sequences of components $\{\, (\alpha_i, \beta_i) \;:\; i \in \mathbb{N} \,\}$ and $\{\, (\sigma_i, \tau_i) \;:\; i \in \mathbb{N} \,\}$ of $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$, respectively, such that $\beta_i \in (\sigma_i, \tau_i)$ and $\tau_i \in (\alpha_{i+1}, \beta_{i+1})$ for all $i \in \mathbb{N}$. Also by construction $(\alpha_i, \beta_i) \cap (\alpha_j, \beta_j) = \emptyset$ and $(\sigma_i, \tau_i) \cap (\sigma_j, \tau_j) = \emptyset$ for all $i \neq j$. Now, there exists $s_1 \in \mathbb{Z}$ such that $f^{s_1}(\Sigma_1) \subseteq (\sigma_1, \tau_1)$ and there exists $h_1 \in \mathcal{F}_V^{<\infty}$ extending h_0 and $t_1 \in \mathbb{Z}$ such that $h_1^{t_1} f^{s_1}(\Sigma_1) \subseteq (\alpha_2, \beta_2)$ and $[\operatorname{dom}(h_1) \cup \operatorname{ran}(h_1)] \setminus [\operatorname{dom}(h_0) \cup \operatorname{ran}(h_0)] \subseteq (\sigma_1, \tau_1)$.

We continue this process obtaining $s_1, t_1, s_2, t_2, \ldots \in \mathbb{Z}$ and $h_1, h_2, \ldots \in \mathcal{F}_V^{<\infty}$ such that h_{i+1} is an extension of h_i and the following hold for all $i \in \mathbb{N}$:

- $f^{s_i} h_{i-1}^{t_{i-1}} \cdots h_1^{t_1} f^{s_1}(\Sigma_1) \subseteq (\sigma_i, \tau_i)$;
- $h_i^{t_i} f^{s_i} \cdots h_1^{t_1} f^{s_1}(\Sigma_1) \subseteq (\alpha_{i+1}, \beta_{i+1})$;
- $[\operatorname{dom}(h_i) \cup \operatorname{ran}(h_i)] \setminus [\operatorname{dom}(h_{i-1}) \cup \operatorname{ran}(h_{i-1})] \subseteq (\sigma_i, \tau_i)$.

As the domain of h_0 is finite, there exists $i \in \mathbb{Z}$ such that (σ_i, τ_i) contains no elements of $\operatorname{dom}(h_0) \cup \operatorname{ran}(h_0)$. Moreover,

$$\operatorname{dom}(h_{i-1}) \cup \operatorname{ran}(h_{i-1}) \subseteq [\operatorname{dom}(h_0) \cup \operatorname{ran}(h_0)] \cup \bigcup_{j=1}^{i-1} (\sigma_j, \tau_j).$$

Hence, by construction, (σ_i, τ_i) also contains no elements of $\operatorname{dom}(h_{i-1}) \cup \operatorname{ran}(h_{i-1})$. So, if

$$k = f^{s_i} h_{i-1}^{t_{i-1}} \cdots h_1^{t_1} f^{s_1} h_{i-1}^{m_j} f^{m_{j-1}} \cdots h_{i-1}^{m_2} f^{m_1},$$

then $k(\Sigma_0) \subseteq (\sigma_i, \tau_i)$ and so $h_{i-1} \cup kpk^{-1}$ is well-defined and an element of $\mathcal{F}_V^{<\infty}$. Finally, if $h \in \mathcal{F}_V \cap [h_{i-1} \cup kpk^{-1}]$, then $\langle\, f, h \,\rangle \cap [p] \neq \emptyset$. Therefore $h \in A_p \cap [g_0]$ and so $A_p \cap [g_0] \neq \emptyset$, as required.

(ii). $\mathbb{R} \setminus U$ *has an unbounded component and* V *is bounded.* Without loss of generality we may assume that (α, ∞) is an unbounded component of $\mathbb{R} \setminus U$. By Lemma 3.1, there exist an extension g_1 of g_0 in $\mathcal{F}_V^{<\infty}$ and $m_1, m_2, \ldots, m_t \in \mathbb{Z}$ such that

$$g_1^{m_j} f^{m_{j-1}} \cdots g_1^{m_2} f^{m_1}(\Sigma_0) \subseteq (\alpha, \infty).$$

Now, there exists $n \in \mathbb{Z}$ such that

$$\min\{f^n g_1^{m_j} \cdots f^{m_1}(\Sigma_0)\} > \max\{\operatorname{dom}(g_1) \cup \operatorname{ran}(g_1) \cup V\}.$$

If g_1 has parity $+1$ on $\mathrm{dom}(g_1) \cap (\max V, \infty)$, then we define

$$g_2 = g_1 \cup f^n g_1^{m_j} \cdots f^{m_1} p (f^n g_1^{m_j} \cdots f^{m_1})^{-1}.$$

Otherwise, define

$$g_2 = g_1 \cup f^n g_1^{m_j} \cdots f^{m_1} p^{-1} (f^n g_1^{m_j} \cdots f^{m_1})^{-1}.$$

Let $h \in \mathcal{F}_V^{<\infty} \cap [g_2]$. Then $\langle\, f, h \,\rangle \cap [p] \neq \emptyset$ and so $A_p \cap [g_0] \neq \emptyset$, as required. □

We next give the proof of Corollary 1.4.

Proof of Corollary 1.4. Let $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be such that $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ is unbounded above and below and let $\mathcal{F} = \{\, g \in \mathrm{Aut}(\mathbb{Q}, \leqslant) \,:\, \mathrm{fix}(\tilde{f}) \cap \mathrm{fix}(\tilde{g}) = \emptyset \,\}$. From the comments at the start of the proof of Theorem 1.1, $\mathcal{D}_f$ is a G_δ subset of $\mathrm{Aut}(\mathbb{Q}, \leqslant)$. Hence we only need to verify that $\mathcal{D}_f \cap \mathcal{F}$ is dense in $\mathcal{F}$. To this end, let $g \in \mathcal{F}^{<\infty}$.

Let $V \subseteq \mathbb{R}$ be closed. Then it is straightforward to verify that $g \in \mathcal{F}_V^{<\infty}$ if and only if $\mathrm{fix}(g) \subseteq V$, $V \cap [\alpha, g(\alpha)] = \emptyset$ if $\alpha < g(\alpha)$, $V \cap [g(\alpha), \alpha] = \emptyset$ if $g(\alpha) < \alpha$, and $V \cap (\alpha, \beta) \neq \emptyset$ for all $\alpha, \beta \in \mathrm{dom}(g)$ with $g(\alpha) < \alpha < \beta < g(\beta)$ or $\alpha < g(\alpha) < g(\beta) < \beta$.

Assume that $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ has an unbounded component. Let $V \subseteq \mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ be any finite subset of $\mathbb{Q}$ satisfying the conditions in the previous paragraph. Then $g \in \mathcal{F}_V^{<\infty}$. Since V is finite, it is certainly bounded and so, by Theorem 1.1(ii), $\mathcal{D}_f \cap \mathcal{F}_V$ is comeagre in $\mathcal{F}_V$. In particular, $\mathcal{D}_f$ is dense in $\mathcal{F}_V$, and so $\mathcal{D}_f \cap [g] \cap \mathcal{F}_V \neq \emptyset$. Therefore $\mathcal{D}_f \cap \mathcal{F}$ is dense in $\mathcal{F}$.

Assume that all the components of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ are bounded. Then, by our assumption that $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ is unbounded above and below, there exists a closed subset V of $\mathbb{R} \setminus \mathrm{fix}(\tilde{f})$ satisfying the above conditions and where V is unbounded above and below. Then $g \in \mathcal{F}_V^{<\infty}$ and all the components of $\mathbb{R} \setminus V$ are bounded. Hence, by Theorem 1.1(i), $\mathcal{D}_f \cap \mathcal{F}_V$ is comeagre in $\mathcal{F}_V$ and, in particular, $\mathcal{D}_f \cap [g] \cap \mathcal{F}_V \neq \emptyset$, as required. □

The following lemma is required in the proof of Theorem 1.3.

Lemma 3.3. *Let $f, g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ be non-identity elements such that $\mathrm{fix}(\tilde{f}) \cap \mathrm{fix}(\tilde{g}) = \emptyset$ and there exists $\alpha \in \mathbb{R}$ such that $\Sigma = (-\infty, \alpha)$ or (α, ∞) is an orbital of g. If $[h] \cap \langle f, g \rangle \neq \emptyset$ for all $h \in \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$ with $\mathrm{dom}(h) \cup \mathrm{ran}(h) \subseteq \Sigma$, then f and g topologically generate $\mathrm{Aut}(\mathbb{Q}, \leqslant)$.*

Proof. Let $k \in \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$ be arbitrary. Assume without loss of generality that $(-\infty, \alpha)$ is an orbital of g. This implies that $\alpha \in \mathrm{fix}(\tilde{g})$. By Lemma 3.1, there exists $p \in \langle\, f, g \,\rangle$ such that $p(\max\{\mathrm{dom}(k) \cup \mathrm{ran}(k)\}) < \alpha$ and so $p(\mathrm{dom}(k) \cup \mathrm{ran}(k)) \subseteq (-\infty, \alpha)$. Thus $h = pkp^{-1}$ satisfies

$$\mathrm{dom}(h) \cup \mathrm{ran}(h) \subseteq (-\infty, \alpha).$$

Hence, by assumption, $[h] \cap \langle\, f, g \,\rangle \neq \emptyset$. Therefore since $p \in \langle\, f, g \,\rangle$, it follows that $[k] \cap \langle\, f, g \,\rangle \neq \emptyset$. We have shown that f and g topologically generate $\mathrm{Aut}(\mathbb{Q}, \leqslant)$, as required. □

Proof of Theorem 1.3. Recall that $f \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ is any non-identity element. We must prove that there exists $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ such that f and g topologically generate $\mathrm{Aut}(\mathbb{Q}, \leqslant)$. If $\mathrm{fix}(\tilde{f}) = \emptyset$, then the result follows by Theorem 1.1(ii).

So, we may assume that $\mathrm{fix}(\tilde{f}) \neq \emptyset$. Then there exists $\beta \in \mathrm{fix}(\tilde{f})$ such that β is the supremum of a non-singleton orbital of $\mathbb{Q} \setminus \mathrm{fix}(\tilde{f})$. Let $\alpha \in \mathbb{Q} \setminus \mathrm{fix}(\tilde{f})$ be any element in the orbital of f of which β is the supremum. Assume without loss of generality that the orbital of f containing α has parity $+1$. We define $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ by induction such that $\mathrm{fix}(\tilde{g}) = \{\alpha\}$. The first step is to let g_0 be any order-preserving bijection on $\mathbb{Q}$ with $(-\infty, \alpha)$ as an orbital, $g_0(\alpha) = \alpha$, $g_0^{-2}(\beta) = f(\alpha)$, and $g_0^{-1}(\beta) = f^2(\alpha)$. It is straightforward to verify that g_0 can be extended to an element $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ such that $\mathrm{fix}(\tilde{g}) = \{\alpha\}$.

Enumerate the functions $h \in \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$ such that $\mathrm{dom}(h) \cup \mathrm{ran}(h) \subseteq (-\infty, \alpha)$ as $h_1, h_2, \ldots$. Assume that for $n \geqslant 0$ the function g_n is defined such that the following hold:

- if $n > 0$, then g_n is an order-preserving bijection extending g_{n-1};
- $\mathrm{dom}(g_n) \cap (\alpha, \infty)$ is a finite subset of $[f(\alpha), f^2(\alpha))$;
- $\mathrm{dom}(g_n) \setminus \mathrm{dom}(g_{n-1}) \subseteq (f^2(\alpha) - 1/n, f^2(\alpha))$;
- $\mathrm{ran}(g_n) \cap (\alpha, \infty) \subseteq [f^2(\alpha), f^{n+2}(\alpha))$;
- $\mathrm{ran}(g_n) \cap (f^{n+1}(\alpha), f^{n+2}(\alpha)) \neq \emptyset$;
- if $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ is any extension of g_n, then $\langle\, f, g \,\rangle \cap [h_m] \neq \emptyset$ for all $m \leqslant n$.

Note that since g_n is an order-preserving bijection and $\mathrm{dom}(g_n) \cap (\alpha, \infty)$ is finite, there exists an extension $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ of g_n with $\mathrm{fix}(\tilde{g}) = \{\alpha\}$.

We now define g_{n+1}. Let $\sigma_n \in (f^2(\alpha) - 1/(n+1), f^2(\alpha))$ be such that $\sigma_n > \max\{\mathrm{dom}(g_n) \cap (f(\alpha), f^2(\alpha))\}$. Note that σ_n exists by the assumption that $\mathrm{dom}(g_n) \cap (\alpha, \infty)$ is finite. Hence $f^{-2}(\sigma_n) \in (f^{-1}(\alpha), \alpha)$ and so there exists $j \in \mathbb{Z}$ such that

$$g_n^j(\mathrm{dom}(h_{n+1})) \subseteq (f^{-2}(\sigma_n), \alpha) \text{ and } g_n^j(\mathrm{ran}(h_{n+1})) \subseteq (f^{-1}(\alpha), \alpha).$$

Hence $f^2 g_n^j(\mathrm{dom}(h_{n+1})) \subseteq (\sigma_n, f^2(\alpha))$ and $f^{n+3} g_n^j(\mathrm{ran}(h_{n+1})) \subseteq (f^{n+2}(\alpha), f^{n+3}(\alpha))$. So, define g_{n+1} to agree with g_n on $\mathrm{dom}(g_n)$ and define

$$g_{n+1}(f^2 g_n^j(\gamma)) = f^{n+3} g_n^j h_{n+1}(\gamma)$$

for all $\gamma \in \mathrm{dom}(h_{n+1})$. Then g_{n+1} is an order-preserving bijection which extends g_n. By construction, $\mathrm{dom}(g_{n+1}) \cap (\alpha, \infty) \subseteq [f(\alpha), f^2(\alpha))$ is finite, $\mathrm{ran}(g_{n+1}) \cap (\alpha, \infty) \subseteq [f^2(\alpha), f^{n+3}(\alpha))$, and $\mathrm{ran}(g_{n+1}) \cap (f^{n+2}(\alpha), f^{n+3}(\alpha)) \neq \emptyset$. Finally, if $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ is any extension of g_{n+1}, then $g^{-j} f^{-n-3} g f^2 g^j \in [h_{n+1}] \cap \langle\, f, g \,\rangle$. Thus $\langle\, f, g \,\rangle \cap [h_m] \neq \emptyset$ for all $m \leqslant n+1$.

Hence, by induction, if $p = \bigcup_{n=0}^{\infty} g_n$, then p is an order-preserving bijection,

$$\mathrm{dom}(p) \cap (\alpha, \infty) \subseteq [f(\alpha), f^2(\alpha)) = [p^{-2}(\beta), p^{-1}(\beta)) \text{ and}$$
$$\mathrm{ran}(p) \cap (\alpha, \infty) \subseteq [f^2(\alpha), \beta) = [p^{-1}(\beta), \beta).$$

Furthermore, since $f^i(\alpha) \longrightarrow \beta$ as $i \longrightarrow \infty$, $\sup\{\mathrm{ran}(p) \cap (\alpha, \infty)\} = \beta$. It follows from the construction of p that there exists an extension $g \in \mathrm{Aut}(\mathbb{Q}, \leqslant)$ of p with $\mathrm{fix}(\tilde{g}) = \{\alpha\}$ and $[h_m] \cap \langle\, f, g \,\rangle \neq \emptyset$ for all m. Therefore, by Lemma 3.3, f and g topologically generate $\mathrm{Aut}(\mathbb{Q}, \leqslant)$, as required. □

Proof of Theorem 1.5. Let $\alpha \in \mathbb{N}$ and define

$$A_\alpha = \{\, (f, g) \in \mathrm{Aut}(\mathbb{Q}, \leqslant) \times \mathrm{Aut}(\mathbb{Q}, \leqslant) \;:\; (\exists \beta > \alpha)\ (f(\beta) = g(\beta) = \beta) \,\}.$$

It is clear that A_α is open in $\mathrm{Aut}(\mathbb{Q}, \leqslant) \times \mathrm{Aut}(\mathbb{Q}, \leqslant)$. We will prove that A_α is dense in $\mathrm{Aut}(\mathbb{Q}, \leqslant) \times \mathrm{Aut}(\mathbb{Q}, \leqslant)$. To this end, let $(g_1, g_2) \in \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty} \times \mathrm{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$.

Let $\sigma \in \mathbb{Q}$ be such that σ is greater than every element of $\operatorname{dom}(g_1) \cup \operatorname{dom}(g_2) \cup \operatorname{ran}(g_1) \cup \operatorname{ran}(g_2)$. Then there are extensions $h_1, h_2 \in \operatorname{Aut}(\mathbb{Q}, \leqslant)$ of g_1, g_2, respectively, such that $h_1(\sigma) = h_2(\sigma) = \sigma$. Hence A_α is dense and open in $\operatorname{Aut}(\mathbb{Q}, \leqslant) \times \operatorname{Aut}(\mathbb{Q}, \leqslant)$, and so $A = \bigcap_{\alpha=1}^{\infty} A_\alpha$ is comeagre in $\operatorname{Aut}(\mathbb{Q}, \leqslant) \times \operatorname{Aut}(\mathbb{Q}, \leqslant)$.

It remains to prove that $\langle f, g \rangle$ is nowhere dense in $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ for all $(f, g) \in A$. Let $h \in \operatorname{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$. Then, since $(f, g) \in A$, there exists $\beta \in \mathbb{Q}$ such that $f(\beta) = g(\beta) = \beta$ and β is larger than every element of $\operatorname{dom}(h) \cup \operatorname{ran}(h)$. Hence if k is defined so that $k = h$ on the domain of h and $k(\beta) = \beta + 1$, then $k \in \operatorname{Aut}(\mathbb{Q}, \leqslant)^{<\infty}$ and $\langle f, g \rangle \cap [k] = \emptyset$. □

It is natural to ask for a necessary and sufficient condition on the disjoint closed subsets U and V of $\mathbb{R}$ for $\mathcal{D}_f \cap \mathcal{F}_V$ to be comeagre in $\mathcal{F}_V$ for all $f \in \mathcal{F}_U$. However, we have been unable to determine such a condition. The following examples go some way towards establishing the necessity of the conditions in Theorem 1.1.

Example 3.4. Let U be a closed subset of $\mathbb{R}$ such that $\mathbb{R} \setminus U$ is bounded, let $V = \emptyset$, and let $f \in \mathcal{F}_U$ be arbitrary. Then there exists $\alpha \in \mathbb{Z}$ such that f is fixed outside the interval $[-\alpha, \alpha]$. Let $h \in \mathcal{F}_V^{<\infty}$ be such that $h(-\alpha - 1) = \alpha + 1$. Then $g \notin \mathcal{D}_f$ for all $g \in \mathcal{F}_V \cap [h]$ and so $\mathcal{D}_f \cap \mathcal{F}_V$ is not dense. In particular, the conclusion of Theorem 1.1 does not hold in this case.

If $U = \emptyset$ and V is a closed subset of $\mathbb{R}$ that is unbounded above and below, then there exists an increasing sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ in V that is unbounded above and below. Let $f \in \mathcal{F}_U$ be such that $f(\alpha_n) = \alpha_{n+1}$ for all $n \in \mathbb{Z}$. Then $\langle f, g \rangle$ is not dense in $\operatorname{Aut}(\mathbb{Q}, \leqslant)$ for all $g \in \mathcal{F}_V$. That is, $\mathcal{F}_V \cap \mathcal{D}_f = \emptyset$, and again the conclusion of Theorem 1.1 does not hold in this case.

4. The random graph - the proofs

In this section we give the proof of Theorem 1.6 and its corollaries in a series of lemmas. Recall that throughout the paper, the $\mathcal{C}$-coloured random graph $R_\mathcal{C}$ is the triple $(\Omega, \mathcal{C}, F)$ where Ω is a countably infinite set, $\mathcal{C} = \{\lambda_1, \ldots\}$ is a finite with more than one element, and F is a function from the two-element subsets of Ω into $\mathcal{C}$.

The following collection of isomorphisms between finite subsets of $R_\mathcal{C}$ is important for the proofs of the results in this section

$$\mathcal{P} = \{\ f \in \operatorname{Aut}(R_\mathcal{C})^{<\infty}\ :\ \operatorname{dom}(f) \cap \operatorname{ran}(f) = \emptyset \text{ and } F\{\alpha, f(\alpha)\} = F\{\beta, f(\beta)\} \\ \text{for all } \ \alpha, \beta \in \operatorname{dom}(f)\ \}.$$

The proof of the next lemma is straightforward but we include it for the sake of completeness.

Lemma 4.1. *Let G be a subgroup of $\operatorname{Aut}(R_\mathcal{C})$ such that $G \cap [f] \neq \emptyset$ for all $f \in \mathcal{P}$. Then G is dense in $\operatorname{Aut}(R_\mathcal{C})$.*

Proof. Let $f \in \operatorname{Aut}(R_\mathcal{C})^{<\infty}$ be arbitrary and $\lambda \in \mathcal{C}$ be fixed. We will show that $G \cap [f] \neq \emptyset$. There exists, by repeated application of the Alice's restaurant property, a finite subset Σ of $\Omega \setminus [\operatorname{dom}(f) \cup \operatorname{ran}(f)]$ such that Σ is isomorphic to $\operatorname{dom}(f)$ (and $\operatorname{ran}(f)$) and $F\{\alpha, \beta\} = F\{\beta, \gamma\} = \lambda$ for all $\alpha \in \operatorname{dom}(f)$, $\beta \in \Sigma$, and $\gamma \in \operatorname{ran}(f)$. If $g : \operatorname{dom}(f) \longrightarrow \Sigma$ is an isomorphism, then $g \in \mathcal{P}$. Also $fg^{-1} : \Sigma \longrightarrow \operatorname{ran}(f)$ is an element of $\mathcal{P}$. Hence, as $f = fg^{-1}g$, $G \cap [g] \neq \emptyset$, and $G \cap [fg^{-1}] \neq \emptyset$, we have that $G \cap [f] \neq \emptyset$, verifying that G is dense in $\mathrm{Aut}(R_{\mathcal{C}})$. □

Next, we describe a general method for constructing extensions of elements of $\mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$ that provides a means of proving Theorem 1.6.

Let

$$\mathcal{G} = \{\, f \in \mathrm{Sym}(\mathbb{N})^{<\infty} \ : \ f \text{ contains no cycle } \},$$

let $f \in \mathcal{G}$, let A be a subset of $\mathrm{dom}(f) \cup \mathrm{ran}(f)$, and let c be a function from the two element subsets of A to $\mathcal{C}$. We say that the system (f, A, c) is *consistent* if $c\{i,j\} = c\{f^l(i), f^l(j)\}$ for all $i, j \in A$ and for all $l \in \mathbb{Z}$ such that $f^l(i), f^l(j) \in A$.

Lemma 4.2. *If (f, A, c) is consistent and $i \in \mathrm{dom}(f) \cup \mathrm{ran}(f)$, then there exists an extension c' of c such that $(f, A \cup \{i\}, c')$ is consistent.*

Proof. If $i \in A$, then setting $c' = c$ the proof is complete.

If $i \notin A$, then define c' as follows. Let $\lambda \in \mathcal{C}$ be fixed and let $j, k \in A \cup \{i\}$ be arbitrary. Then define $c'\{j,k\} = c\{f^l(j), f^l(k)\}$ if there exists $l \in \mathbb{Z}$ such that $f^l(j), f^l(k) \in A$ and define $c'\{j,k\} = \lambda$ otherwise. Since (f, A, c) is consistent, it follows that c' is well-defined.

To verify that $(f, A \cup \{i\}, c')$ is consistent, let $j, k \in A \cup \{i\}$ with $j \neq k$ and let $l \in \mathbb{Z}$ be arbitrary. If there exists $m \in \mathbb{Z}$ such that $f^m(j), f^m(k) \in A$, then from the definition of c' on $\{j,k\}$ and $\{f^l(j), f^l(k)\}$

$$c'\{j,k\} = c\{f^m(j), f^m(k)\} = c'\{f^l(j), f^l(k)\}.$$

On the other hand, if $f^m(j) \notin A$ or $f^m(k) \notin A$ for all $m \in \mathbb{Z}$, then

$$c'\{j,k\} = \lambda = c'\{f^l(j), f^l(k)\},$$

as required. □

The following corollary is obtained by repeatedly applying Lemma 4.2.

Corollary 4.3. *If (f, A, c) is consistent, then there exists an extension c' of c such that $(f, \mathrm{dom}(f) \cup \mathrm{ran}(f), c')$ is consistent.*

The next lemma relates consistent systems (f, A, c) and elements of $\mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$.

Lemma 4.4. *Let Σ be a finite subset of Ω, let (f, A, c) be a consistent system, and let $\Psi : A \longrightarrow \Omega \setminus \Sigma$ be an injective function such that $c\{i,j\} = F\{\Psi(i), \Psi(j)\}$ for all $i, j \in A$. Then there exists an extension $\Phi : \mathrm{dom}(f) \cup \mathrm{ran}(f) \longrightarrow \Omega \setminus \Sigma$ of Ψ such that $\Phi f \Phi^{-1} \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$.*

Proof. Let $[\mathrm{dom}(f) \cup \mathrm{ran}(f)] \setminus A = \{i_1, i_2, \ldots, i_n\}$ and let Φ agree with Ψ on A. We complete the definition of Φ as follows. By Corollary 4.3, there exists an extension c' of c such that $(f, \mathrm{dom}(f) \cup \mathrm{ran}(f), c')$ is consistent. By the Alice's restaurant property, there exists $\alpha_1 \in \Omega \setminus [\Sigma \cup \mathrm{ran}(\Psi)]$ such that $F\{\alpha_1, \Phi(j)\} = c'\{i_1, j\}$ for all $j \in A$. So, we may define $\Phi(i_1) = \alpha_1$. Assuming that $\Phi(i_1), \ldots, \Phi(i_{m-1})$ are defined for all m such that $2 \leqslant m < n$,then $\Phi(i_m)$ is defined to be any $\alpha_m \in \Omega \setminus [\Sigma \cup \mathrm{dom}(\Psi) \cup \mathrm{ran}(\Psi) \cup \{\Phi(i_1), \ldots, \Phi(i_{m-1})\}]$ such that $F\{\alpha_m, \Phi(j)\} = c'\{i_m, j\}$ for all $j \in A \cup \{i_1, \ldots, i_{m-1}\}$. Such an element α_m exists, again by the Alice's restaurant property.

Let $\alpha, \beta \in \operatorname{dom}(\Phi f \Phi^{-1}) = \Phi(\operatorname{dom}(f))$ be arbitrary. Then

$$\begin{aligned} F\{\alpha, \beta\} &= c'\{\Phi^{-1}(\alpha), \Phi^{-1}(\beta)\} = c'\{f\Phi^{-1}(\alpha), f\Phi^{-1}(\beta)\} \\ &= F\{\Phi f \Phi^{-1}(\alpha), \Phi f \Phi^{-1}(\beta)\}, \end{aligned}$$

and so $\Phi f \Phi^{-1} \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$. □

Lemma 4.5. *Let Σ be a finite subset of Ω, let $f \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$ be such that f has no cycles and $[\operatorname{dom}(f) \cup \operatorname{ran}(f)] \cap \Sigma = \emptyset$, let $\alpha \in \operatorname{dom}(f) \setminus \operatorname{ran}(f)$, and let $\beta \in \Omega \setminus [\operatorname{dom}(f) \cup \operatorname{ran}(f) \cup \Sigma]$. Then there exists $g \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$ extending f such that g has no cycles, $[\operatorname{dom}(g) \cup \operatorname{ran}(g)] \cap \Sigma = \emptyset$, f and g have the same number of partial orbits, $\beta \in \operatorname{Orb}(g, \alpha)$, and $\operatorname{Orb}(g, \gamma) = \operatorname{Orb}(f, \gamma)$ for all $\gamma \in [\operatorname{dom}(f) \cup \operatorname{ran}(f)] \setminus \operatorname{Orb}(f, \alpha)$.*

Proof. By Proposition 2.3(ii), we may assume without loss of generality that all the partial orbits of f have length equal to some $m \in \mathbb{N}$. Let Ψ be any bijection from a subset A of $\mathbb{N}$ to $\operatorname{dom}(f) \cup \operatorname{ran}(f) \cup \{\beta\}$, let $h = \Psi^{-1} f \Psi \in \mathcal{G}$, let $i_0 = \Psi^{-1}(\alpha)$, and let $i_1, i_2 \ldots, i_{m-1} \in \mathbb{N} \setminus A$ be arbitrary distinct elements. Extend h to k so that $k(h^{m-1}(i_0)) = i_1$, $k(i_j) = i_{j+1}$ for all j such that $1 \leqslant j \leqslant m-2$,and $k(i_{m-1}) = \Psi^{-1}(\beta)$. Define c on the two element subsets $\{i, j\}$ of A by $c\{i, j\} = F\{\Psi(i), \Psi(j)\}$.

We will prove that (k, A, c) is consistent. Let $i, j \in A$ and let $l \in \mathbb{Z} \setminus \{0\}$ be such that $k^l(i), k^l(j) \in A$. From the definitions of Ψ and k, it follows that $i, j, k^l(i), k^l(j) \in \Psi^{-1}(\operatorname{dom}(f) \cup \operatorname{ran}(f))$. In particular, none of $i, j, k^l(i), k^l(j)$ equals $\Psi^{-1}(\beta)$. As $f \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$,

$$\begin{aligned} c\{i, j\} &= F\{\Psi(i), \Psi(j)\} = F\{f^l\Psi(i), f^l\Psi(j)\} \\ &= c\{\Psi^{-1} f^l \Psi(i), \Psi^{-1} f^l \Psi(j)\} = c\{h^l(i), h^l(j)\} = c\{k^l(i), k^l(j)\} \end{aligned}$$

and so (k, A, c) is consistent.

By Lemma 4.4, we may extend Ψ to $\Phi : \operatorname{dom}(k) \cup \operatorname{ran}(k) \longrightarrow \Omega \setminus \Sigma$ so that $\Phi k \Phi^{-1} \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$. Then $g = \Phi k \Phi^{-1}$ is the desired extension of f. □

Let Σ be a finite subset of Ω. Then in the introduction we defined

$$\mathcal{I}_\Sigma = \{\ f \in \mathcal{I}\ :\ \Sigma \text{ is a set of representatives of all the orbits of } f\ \}.$$

In the next lemma we give a characterization of those elements in $\operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$ that lie in $\mathcal{I}_\Sigma^{<\infty}$.

Lemma 4.6. *Let Σ be a finite subset of Ω and let $f \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$ be such that f has at most $|\Sigma|$ partial orbits and f has no cycles. Then $f \in \mathcal{I}_\Sigma^{<\infty}$ if and only if $|\operatorname{Orb}(f, \alpha) \cap \Sigma| \leqslant 1$ for all $\alpha \in \operatorname{dom}(f) \cup \operatorname{ran}(f)$.*

Proof. ($\Rightarrow$) This implication is obvious.

($\Leftarrow$) We have to prove that $[f] \cap \mathcal{I}_\Sigma \neq \emptyset$. By Proposition 2.3(i) and Lemma 4.5, there exists $f_1 \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$ extending f such that f_1 has $n = |\Sigma|$ partial orbits, $|\operatorname{Orb}(f_1, \alpha) \cap \Sigma| = 1$ for all $\alpha \in \operatorname{dom}(f_1) \cup \operatorname{ran}(f_1)$. Let $\operatorname{dom}(f_1) \setminus \operatorname{ran}(f_1) = \{\alpha_0, \ldots, \alpha_{n-1}\}$.

We define a sequence of extensions $f_2, f_3, \ldots$ of f_1 whose union lies in $\mathcal{I}_\Sigma$ by induction as follows. Let $\Omega = \{\delta_1, \delta_2, \ldots\}$. At step $i > 1$, let $r \equiv i \pmod{n}$ be such that $0 \leqslant r \leqslant n-1$ and let $\beta_i = \delta_k$, where $k = \min\{\ j \in \mathbb{N}\ :\ \delta_j \notin \operatorname{dom}(f_i) \cup \operatorname{ran}(f_i)\ \}$. Then by Lemma 4.5 applied to $\emptyset$, f_i, α_r, and β_i, we obtain $g_i \in$

$\mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$ extending f_i such that $\beta_i \in \mathrm{Orb}(g_i, \alpha_r)$ and $\mathrm{Orb}(g_i, \gamma) = \mathrm{Orb}(f_i, \gamma)$ for all $\gamma \in [\mathrm{dom}(g_i) \cup \mathrm{ran}(g_i)] \setminus \mathrm{Orb}(g_i, \alpha_r)$.

To conclude step i, let $\gamma_i = \delta_l$ where $l = \min\{ j \in \mathbb{N} : \delta_j \notin \mathrm{dom}(g_i) \cup \mathrm{ran}(g_i) \}$. Then as above applying Lemma 4.5 to $\emptyset$, g_i^{-1}, α_r, and γ_i we obtain $f_{i+1} \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$ extending g_i such that $\gamma_i \in \mathrm{Orb}(f_{i+1}, \alpha_r)$ and $\mathrm{Orb}(f_{i+1}, \gamma) = \mathrm{Orb}(g_i, \gamma)$ for all $\gamma \in [\mathrm{dom}(f_{i+1}) \cup \mathrm{ran}(f_{i+1})] \setminus \mathrm{Orb}(f_{i+1}, \alpha_r)$.

If $h = \bigcup_{n=1}^{\infty} f_i$, then $h \in \mathcal{I}_\Sigma \cap [f]$, as required. □

The next combinatorial lemma is required to prove the lemma after it, which is the main step in the proof of Theorem 1.6.

Lemma 4.7. *Let $m, r \in \mathbb{N}$ and let $N = \{0, 1, \ldots, m-1, k(1), k(2), \ldots, k(2r)\}$ be such that $k(1) > 4m$, $k(2i) - k(2i-1) = 2m$, and $k(2i+1) - k(2i) > 2\,k(2i)$. If $i, j \in N$ with $i < j$ and $l \in \mathbb{Z} \setminus \{0\}$ are such that $i + l, j + l \in N$, then one of the following holds:*

- (i) $i, j, i+l, j+l \in \{0, 1, \ldots, m-1\}$;
- (ii) $i = k(2s-1)$, $j = k(2s)$, $i + l = k(2t-1)$, *and* $j + l = k(2t)$ *for some* s, t *such that* $1 \leqslant s, t \leqslant 2r$;
- (iii) $i = k(2s-1)$, $j = k(2t-1)$, *and* $l = 2m$ *for some* s, t *such that* $1 \leqslant s, t \leqslant 2r$.

Proof. We first make some observations about N. If $p, q \in N$ and one of p and q is an element of $\{k(1), k(2), \ldots, k(2r)\}$, then $|p - q| \geqslant 2m$. Also note that if $p, q \in \{k(1), k(2), \ldots, k(2r)\}$, $p < q$, then

$$q - p = 2m \text{ or } q - p > 2p. \tag{2}$$

Now we proceed to the proof of the lemma. There are three cases to consider.

If $i, j \in \{0, 1, \ldots, m-1\}$, then $|i-j| \leqslant m-1$. So, if $i+l \in \{k(1), k(2), \ldots, k(2r)\}$ or $j + l \in \{k(1), k(2), \ldots, k(2r)\}$, then from above $|i - j| = |(i + l) - (j + l)| \geqslant 2m$, a contradiction. Thus $i + l, j + l \in \{0, 1, \ldots, m-1\}$ and (i) is satisfied.

Assume that $i \in \{0, 1, \ldots, m-1\}$ and $j \in \{k(1), k(2), \ldots, k(2r)\}$. Then $2m < |i - j| \leqslant j$. If $l > 0$, then $j + l \in \{k(1), k(2), \ldots, k(r)\}$. Applying (2) to j and $j + l$, it follows that $l = 2m$ or $l > 2j$. If $l = 2m$, then $2m \leqslant i + l < 3m < k(1)$ and so $i + l \notin N$. If $l > 2j$, then $i + l \in \{k(1), k(2), \ldots, k(2r)\}$ and so again by (2) either $|j-i| = |(j+l)-(i+l)| = 2m$ or $|j-i| = |(j+l)-(i+l)| > 2(i+l) > j$. In either case, we obtain a contradiction. If $l < 0$, then $l \geq -m+1$ and hence $j + l \notin N$. It follows that for all $l \in \mathbb{Z} \setminus \{0\}$, either $i + l \notin N$ or $j + l \notin N$.

Assume that $i, j \in \{k(1), k(2), \ldots, k(2r)\}$ and $i < j$. If $l > 0$, then $i, j, i + l, j + l \in \{k(1), k(2), \ldots, k(2r)\}$ and applying (2) to j and $j + l$, it follows that $l = 2m$ or $l > 2j$. If $l = 2m$ and either $i = k(2s)$ or $j = k(2s)$ for some s such that $1 \leqslant s \leqslant r$,then $i + l$ or $j + l \notin N$. Hence $i = k(2s-1)$ and $j = k(2t-1)$ for some s, t such that $1 \leqslant s, t \leqslant 2r$, and (iii) is satisfied. If $l > 2j$ and (ii) does not hold, then $|i - j| \neq 2m$. Then, it follows, by (2) applied to $i + l$ and $j + l$, that $|i - j| = |(i + l) - (j + l)| > 2(i + l) > 2j$, a contradiction. If $l < 0$ and $i + l, j + l \in N$, then it can be verified that $i + l, j + l \in \{k(1), k(2), \ldots, k(2r)\}$. An argument symmetric to the one just given shows that (ii) or (iii) holds in this situation, completing the proof of the lemma. □

The following lemma provides the critical step in the proof of Theorem 1.6.

Lemma 4.8. *Let Σ be a finite subset of Ω, let $f \in \mathcal{I}_{\Sigma}^{<\infty}$, and let $g \in \mathcal{P}$ be such that $[\Sigma \cup \operatorname{dom}(f) \cup \operatorname{ran}(f)] \cap [\operatorname{dom}(g) \cup \operatorname{ran}(g)] = \emptyset$. Then there exist $h \in \mathcal{I}_{\Sigma}^{<\infty}$ and $j \in \mathbb{N}$ such that h extends f and h^j extends g.*

Proof. Applying Proposition 2.3(i), we may assume that f has exactly $|\Sigma|$ partial orbits, no cycles, no partial orbit of f contains more than one element of Σ and $[\operatorname{dom}(f) \cup \operatorname{ran}(f)] \cap [\operatorname{dom}(g) \cup \operatorname{ran}(g)] = \emptyset$. Then, applying Lemma 4.5 we may assume that f has exactly $|\Sigma|$ partial orbits, no cycles, each partial orbit of f contains exactly one point of Σ, and $[\operatorname{dom}(f) \cup \operatorname{ran}(f)] \cap [\operatorname{dom}(g) \cup \operatorname{ran}(g)] = \emptyset$. Finally, applying Proposition 2.3(ii), we may assume that f has exactly $|\Sigma|$ partial orbits each of length m, for some $m \in \mathbb{N}$, no cycles, each partial orbit contains exactly one element of Σ, and $[\operatorname{dom}(f) \cup \operatorname{ran}(f)] \cap [\operatorname{dom}(g) \cup \operatorname{ran}(g)] = \emptyset$. Lemma 4.6 guarantees that $f \in \mathcal{I}_{\Sigma}^{<\infty}$. Hence it suffices to prove the lemma for the function f as just described.

Let $\operatorname{dom}(g) = \{\sigma_1, \ldots, \sigma_r\}$, let $\{k(1), k(2), \ldots, k(2r)\}$ be any subset of $\mathbb{N}$ satisfying the hypothesis of Lemma 4.7, let Ψ be a bijection from a subset A of $\mathbb{N}$ to $\operatorname{dom}(f) \cup \operatorname{ran}(f) \cup \operatorname{dom}(g) \cup \operatorname{ran}(g)$, let $h_0 = \Psi^{-1} f \Psi$, let $i_0 \in \operatorname{dom}(h_0) \setminus \operatorname{ran}(h_0)$ be an arbitrary but fixed element, and let $\{i_m, i_{m+1}, \ldots, i_{k(2r)}\}$ be distinct elements of $\mathbb{N} \setminus [\operatorname{dom}(h_0) \cup \operatorname{ran}(h_0)]$ such that $\Psi(i_{k(2i-1)}) = \sigma_i$ and $\Psi(i_{k(2i)}) = g(\sigma_i)$ for all i such that $1 \leqslant i \leqslant r$. Extend h_0 to $h_1 \in \mathcal{G}$ so that $h_1(h_0^{m-1}(i_0)) = i_m$ and $h_1(i_j) = i_{j+1}$ for all j such that $m \leqslant j < k(2r)$. Define c from the two element subsets of A to $\mathcal{C}$ by $c\{a, b\} = F\{\Psi(a), \Psi(b)\}$ for all $a, b \in A$.

We will prove that (h_1, A, c) is consistent. Let $a, b \in A$ be distinct and let $l \in \mathbb{Z}$ be such that $c\{h_1^l(a), h_1^l(b)\}$ is defined. Then there exist $u, v \in \operatorname{dom}(h_1) \setminus \operatorname{ran}(h_1)$ and $i, j \in \mathbb{Z}$ such that $a = h_1^i(u)$ and $b = h_1^j(v)$.

Let us first consider the situation when $i \neq j$. Without loss of generality assume that $i < j$. By Lemma 4.7 applied to $N = \operatorname{dom}(h_1)$, one of the following holds:

(i) $i, j, i + l, j + l \in \{0, 1, \ldots, m - 1\}$;
(ii) $i = k(2s - 1)$, $j = k(2s)$, $i + l = k(2t - 1)$, and $j + l = k(2t)$ for some s, t such that $1 \leqslant s, t \leqslant r$;
(iii) $i = k(2s - 1)$, $j = k(2t - 1)$, and $l = 2m$ for some s, t such that $1 \leqslant s, t \leqslant r$.

If (i) holds, then $a, b, h_1^l(a), h_1^l(b) \in \Psi^{-1}(\operatorname{dom}(f) \cup \operatorname{ran}(f)) = \operatorname{dom}(h_0) \cup \operatorname{ran}(h_0)$. Hence, as $f \in \operatorname{Aut}(R_{\mathcal{C}})^{<\infty}$,

$$\begin{aligned} c\{a, b\} &= F\{\Psi(a), \Psi(b)\} = F\{f^l \Psi(a), f^l \Psi(b)\} = c\{\Psi^{-1} f^l \Psi(a), \Psi^{-1} f^l \Psi(b)\} \\ &= c\{h_0^l(a), h_0^l(b)\} = c\{h_1^l(a), h_1^l(b)\}. \end{aligned}$$

If (ii) holds, then $u = v = i_0$. Hence

$$\begin{aligned} c\{a, b\} &= c\{h_1^i(i_0), h_1^j(i_0)\} = c\{i_{k(2s-1)}, i_{k(2s)}\} = F\{\Psi(i_{k(2s-1)}), \Psi(i_{k(2s)})\} \\ &= F\{\sigma_s, g(\sigma_s)\} \end{aligned}$$

and likewise

$$c\{h_1^l(a), h_1^l(b)\} = c\{h_1^{i+l}(i_0), h_1^{j+l}(i_0)\} = c\{i_{k(2t-1)}, i_{k(2t)}\} = F\{\sigma_t, g(\sigma_t)\}.$$

Since $g \in \mathcal{P}$, it follows that $F\{\sigma_s, g(\sigma_s)\} = F\{\sigma_t, g(\sigma_t)\}$, verifying that $c\{a, b\} = c\{h_1^l(a), h_1^l(b)\}$.

If (iii) holds, then again $u = v = i_0$. It follows that

$$\begin{aligned} c\{a,b\} &= c\{h_1^i(i_0), h_1^j(i_0)\} = c\{i_{k(2s-1)}, i_{k(2t-1)}\} = F\{\Psi(i_{k(2s-1)}), \Psi(i_{k(2t-1)})\} \\ &= F\{\sigma_s, \sigma_t\} \end{aligned}$$

and

$$\begin{aligned} c\{h_1^l(a), h_1^l(b)\} &= c\{h_1^{i+l}(i_0), h_1^{i+l}(i_0)\} = c\{i_{2m+k(2s-1)}, i_{2m+k(2t-1)}\} \\ &= c\{i_{k(2s)}, i_{k(2t)}\} = F\{\Psi(i_{k(2s)}), \Psi(i_{k(2t)})\} = F\{g(\sigma_s), g(\sigma_t)\}. \end{aligned}$$

As $g \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$, we have that $F\{\sigma_s, \sigma_t\} = F\{g(\sigma_s), g(\sigma_t)\}$, verifying that $c\{a,b\} = c\{h_1^l(a), h_1^l(b)\}$.

Let us finally consider the situation when $i = j$. As $a \neq b$, we have that $u \neq v$ and hence one of a, b is not in $\mathrm{Orb}(h_1, i_0)$. Thus $i, j \in \{0, 1, \dots, m-1\}$. Since the only partial orbit of h_1 with more than m elements is $\mathrm{Orb}(h_1, i_0)$, it follows that $i + l, j + l \in \{0, 1, \dots, m-1\}$ also. As $f \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$, it follows, by the same argument used when $i \neq j$ and (i) held above, that $c\{h_1^l(a), h_1^l(b)\} = c\{a,b\}$. We have shown that (h_1, A, c) is consistent.

It follows from Lemma 4.4 that there exists an extension $\Phi : \mathrm{dom}(h_1) \cup \mathrm{ran}(h_1) \longrightarrow \Omega$ of Ψ such that $h = \Phi h_1 \Phi^{-1} \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$. But by construction, h_1, and hence h, has $|\Sigma|$ partial orbits and no cycles and so, by Lemma 4.6, $h \in \mathcal{I}_\Sigma^{<\infty}$. By construction, h extends f and h^{2m} extends g. □

We need one more lemma before we can give the proof of Theorem 1.6.

Lemma 4.9. *Let $f \in \mathrm{Aut}(R_{\mathcal{C}})$ be a non-identity element, let $p \in \mathcal{P}$, let Σ be a finite subset of Ω, and let $g \in \mathcal{I}_\Sigma^{<\infty}$. Then there exists $h \in \mathcal{I}_\Sigma^{<\infty}$ extending g such that $\langle f, k \rangle \cap [p] \neq \emptyset$ for all $k \in [h] \cap \mathcal{I}_\Sigma$.*

Proof. As $g \in \mathcal{I}_\Sigma^{<\infty}$, g has an extension in $\mathcal{I}_\Sigma^{<\infty}$ whose domain contains $\mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \Sigma$ and, moreover, has $n = |\Sigma|$ partial orbits of equal length. Hence, it suffices to prove the lemma for such g. To this end, let g be as described, let $\mathrm{dom}(g) \setminus \mathrm{ran}(g) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and let m be the length of each partial orbit of g. We will extend each of $\mathrm{Orb}(g, \alpha_1), \mathrm{Orb}(g, \alpha_2), \dots, \mathrm{Orb}(g, \alpha_n)$ in a particular fashion by m elements using Proposition 2.3(ii).

We proceed by performing a finite recursion. Let $g_0 = g$ and $\Gamma_0 = \mathrm{dom}(g_0) \cup \mathrm{ran}(g_0)$. Applying Proposition 2.3(ii) to $f \in \mathrm{Aut}(R_{\mathcal{C}})$, $g_0 \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$, the finite set

$$\Gamma_0 \cup f(\Gamma_0) \cup f^{-1}(\Gamma_0) \subseteq \Omega$$

and to $g_0^{m-1}(\alpha_1) \in \mathrm{ran}(g_0) \setminus \mathrm{dom}(g_0)$, we obtain a one point extension $g_1 \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$ of g_0 so that $g_1(g_0^{m-1}(\alpha_1)) \notin \Gamma_0 \cup f(\Gamma_0) \cup f^{-1}(\Gamma_0) \cup \mathrm{fix}(f)$. Note that by Lemma 4.6, $g_1 \in \mathcal{I}_\Sigma^{<\infty}$.

At the second step, let $\Gamma_1 = \mathrm{dom}(g_1) \cup \mathrm{ran}(g_1)$. Applying Proposition 2.3(ii) to f, the finite set

$$\Gamma_1 \cup f(\Gamma_1) \cup f^{-1}(\Gamma_1),$$

and to g_1, we obtain a one point extension $g_2 \in \mathrm{Aut}(R_{\mathcal{C}})^{<\infty}$ of g_1 so that $g_2(g_1^m(\alpha_1)) \notin \Gamma_1 \cup f(\Gamma_1) \cup f^{-1}(\Gamma_1) \cup \mathrm{fix}(f)$. Again, $g_2 \in \mathcal{I}_\Sigma^{<\infty}$ follows from Lemma 4.6.

Repeat this process, so that at step m we obtain $g_m \in \mathcal{I}_\Sigma^{<\infty}$, an extension of g_{m-1}, such that $g_m^{2m-1}(\alpha_1) \notin \Gamma_{m-1} \cup f(\Gamma_{m-1}) \cup f^{-1}(\Gamma_{m-1}) \cup \text{fix}(f)$, where $\Gamma_{m-1} = \text{dom}(g_{m-1}) \cup \text{ran}(g_{m-1})$.

At step $m+1$, let $\Gamma_m = \text{dom}(g_m) \cup \text{ran}(g_m)$. Note that $g_m^{m-1}(\alpha_2) = g^{m-1}(\alpha_2)$ ($\in \text{ran}(g_m) \setminus \text{dom}(g_m)$) by construction. Applying Proposition 2.3(ii) to f, the finite set

$$\Gamma_m \cup f(\Gamma_m) \cup f^{-1}(\Gamma_m),$$

and to g_m, we obtain a one point extension $g_{m+1} \in \text{Aut}(R_{\mathcal{C}})^{<\infty}$ of g_m so that $g_{m+1}(g_m^{m-1}(\alpha_2)) \notin \Gamma_m \cup f(\Gamma_m) \cup f^{-1}(\Gamma_m) \cup \text{fix}(f)$. As before, $g_{m+1} \in \mathcal{I}_\Sigma^{<\infty}$ follows.

Repeat this process a total of mn times, to obtain $t = g_{mn} \in \mathcal{I}_\Sigma^{<\infty}$. Observe that by the construction if $1 \leq i \leq n$ and $m \leq j \leq 2m-1$, then

$$t^j(\alpha_i) \notin \Gamma_{m(i-1)+j-m} \cup f(\Gamma_{m(i-1)+j-m}) \cup f^{-1}(\Gamma_{m(i-1)+j-m}) \cup \text{fix}(f).$$

We next show that if $1 \leq i \leq n$ and $m \leq j \leq 2m-1$, then $f(t^j(\alpha_i)) \notin \text{dom}(t) \cup \text{ran}(t)$. If $i' < i$ or ($i' = i$ and $j' < j$), then $f(t^j(\alpha_i)) \neq t^{j'}(\alpha_{i'})$ because $t^j(\alpha_i) \notin f^{-1}(\Gamma_{m(i-1)+j-m})$ and $t^{j'}(\alpha_{i'}) \in \Gamma_{m(i-1)+j-m}$. If $i = i'$ and $j = j'$, then $f(t^j(\alpha_i)) \neq t^{j'}(\alpha_{i'})$ because $t^j(\alpha_i) \notin \text{fix}(f)$. Finally, if $i' > i$ or ($i' = i$ and $j' > j$), then $f(t^j(\alpha_i)) \neq t^{j'}(\alpha_{i'})$ because $t^{j'}(\alpha_{i'}) \notin f(\Gamma_{m(i'-1)+j'-m})$ and $t^j(\alpha_i) \in \Gamma_{m(i'-1)+j'-m}$.

It follows that $q = ft^m pt^{-m} f^{-1}$ satisfies $[\text{dom}(q) \cup \text{ran}(q)] \cap [\text{dom}(t) \cup \text{ran}(t)] = \emptyset$. Moreover, as $p \in \mathcal{P}$, t is a finite isomorphism and f is an automorphism of $R_{\mathcal{C}}$, for all $\alpha, \beta \in \Omega$ with $t^{-m} f^{-1}(\alpha), t^{-m} f^{-1}(\beta) \in \text{dom}(p)$,

$$\begin{aligned} F\{\alpha, q(\alpha)\} &= F\{\alpha, ft^m pt^{-m} f^{-1}(\alpha)\} = F\{t^{-m} f^{-1}(\alpha), pt^{-m} f^{-1}(\alpha)\} \\ &= F\{t^{-m} f^{-1}(\beta), pt^{-m} f^{-1}(\beta)\} = F\{\beta, ft^m pt^{-m} f^{-1}(\alpha)\} \\ &= F\{\beta, q(\beta)\} \end{aligned}$$

and so $q \in \mathcal{P}$.

Now, $t \in \mathcal{I}_\Sigma^{<\infty}$, $q \in \mathcal{P}$,

$$[\text{dom}(t) \cup \text{ran}(t)] \cap [\text{dom}(q) \cup \text{ran}(q)] = \emptyset,$$

and $[\text{dom}(q) \cup \text{ran}(q)] \cap \Sigma = \emptyset$ as $\Sigma \subseteq \text{dom}(t)$. Hence, by Lemma 4.8, there exists $h \in \mathcal{I}_\Sigma^{<\infty}$ and $j \in \mathbb{N}$ such that h extends t (and hence g) and h^j extends q.

Therefore if $k \in [h] \cap \mathcal{I}_\Sigma$, then $k^{-m} f^{-1} k^j f k^m \in [k^{-m} f^{-1} q f k^m] = [p]$, verifying that $\langle f, k \rangle \cap [p] \neq \emptyset$, as required. □

Proof of Theorem 1.6. We must prove that $\mathcal{D}_f \cap \mathcal{I}_\Sigma$ is comeagre in $\mathcal{I}_\Sigma$. Let $p \in \mathcal{P}$ and let $A_p = \{ k \in \mathcal{I}_\Sigma : \langle f, k \rangle \cap [p] \neq \emptyset \}$. We will prove that A_p is dense and open in $\mathcal{I}_\Sigma$.

Let $g \in \mathcal{I}_\Sigma^{<\infty}$ be arbitrary. Then, by Lemma 4.9, there exists $h \in \mathcal{I}_\Sigma^{<\infty}$ extending g such that $\langle f, k \rangle \cap [p] \neq \emptyset$ for all $k \in [h] \cap \mathcal{I}_\Sigma$. So, if $k \in [h] \cap \mathcal{I}_\Sigma$, then $k \in A_p \cap [g]$ and so $A_p \cap [g] \neq \emptyset$. That is, A_p is dense in $\mathcal{I}_\Sigma$.

On the other hand, if $k \in A_p$ is arbitrary, then $\langle f, k \rangle \cap [p] \neq \emptyset$. Hence, as in Lemma 2.4, there exists a finite subset Λ of Ω such that for all $h \in [k_{|\Lambda}] \cap \mathcal{I}_\Sigma$

$$\langle f, h \rangle \cap [p] \neq \emptyset.$$

In other words, $k \in [k_{|\Lambda}] \cap \mathcal{I}_\Sigma \subseteq A_p$, verifying that A_p is open relative to $\mathcal{I}_\Sigma$.

Clearly, $\mathcal{D}_f \cap \mathcal{I}_\Sigma \subseteq \bigcap_{p\in\mathcal{P}} A_p$. Conversely, if $g \in \bigcap_{p\in\mathcal{P}} A_p$, then $\langle f, g \rangle \cap [p] \neq \emptyset$ for all $p \in \mathcal{P}$. So, by Lemma 4.1, $\langle f, g \rangle$ is dense in $\mathrm{Aut}(R_\mathcal{C})$ and so $g \in \mathcal{D}_f \cap \mathcal{I}_\Sigma$. Thus

$$\bigcap_{p\in\mathcal{P}} A_p = \mathcal{D}_f \cap \mathcal{I}_\Sigma$$

and it follows that $\mathcal{D}_f \cap \mathcal{I}_\Sigma$ is comeagre in $\mathcal{I}_\Sigma$. □

The following lemma is required in the proof of Corollary 1.7.

Lemma 4.10. *Let $f \in \mathcal{I}^{<\infty}$ and let $m \in \mathbb{N}$. Then there exists $g \in \mathcal{I}^{<\infty}$ extending f such that every $h \in [g] \cap \mathcal{I}$ has at least m orbits.*

Proof. Let $y_1, y_2, \ldots, y_m$ be distinct elements of $\{0,1\}^m$. Then using the Alice's restaurant property we may find m^2+m distinct elements $\alpha_{1,1}, \alpha_{2,1}, \ldots, \alpha_{m+1,1}, \ldots, \alpha_{m+1,m} \notin \mathrm{dom}(f) \cup \mathrm{ran}(f)$ such that the following hold for all $1 \leq i < i' \leq m+1$ and $1 \leq k \leq m$:

- $F\{\alpha_{i,k}, \alpha_{i',k}\} = \lambda_2$ if $y_k(i'-i) = 1$;
- $F\{\alpha_{i,k}, \alpha_{i',k}\} = \lambda_1$ if $y_k(i'-i) = 0$;
- $F\{\alpha_{i,k}, \beta\} = \lambda_1$ if $\beta = \alpha_{j,k'}$ and $k' \neq k$ or $\beta \in \mathrm{dom}(f) \cup \mathrm{ran}(f)$.

We define g to be the extension of f where $g(\alpha_{i,k}) = \alpha_{i+1,k}$ for all $1 \leq i < m+1$ and $1 \leq k \leq m$. It is routine to verify that $g \in \mathcal{I}^{<\infty}$.

Let $h \in \mathcal{I} \cap [g]$ be arbitrary. We will prove that $\mathrm{Orb}(h, \alpha_{1,j}) \neq \mathrm{Orb}(h, \alpha_{1,k})$ for all j, k such that $1 \leqslant j < k \leqslant m$.Seeking a contradiction assume that there exist $j < k$ such that $\alpha_{1,j} \in \mathrm{Orb}(h, \alpha_{1,k})$. This implies that there exists $r \in \mathbb{Z}$ such that

$$h^r(\alpha_{i,j}) = \alpha_{i,k}$$

for all i such that $1 \leqslant i \leqslant m+1$. But $y_j \neq y_k$ and so there exists l such that $1 \leqslant l \leqslant m$ and $y_j(l) \neq y_k(l)$.Without loss of generality assume that $y_j(l) = 0$ and $y_k(l) = 1$. Hence if $1 \leqslant i < i' \leqslant m+1$ such that $i' - i = l$, then

$$F\{\alpha_{i,j}, \alpha_{i',j}\} = \lambda_1 \neq \lambda_2 = F\{\alpha_{i,k}, \alpha_{i',k}\} = F\{h^r(\alpha_{i,j}), h^r(\alpha_{i',j})\},$$

a contradiction. Thus h has at least m orbits. □

Proof of Corollary 1.7. We must show that $\mathcal{D}_f \cap \mathcal{I}$ is comeagre in $\mathcal{I}$. Let $p \in \mathcal{P}$ and let $A_p = \{\, k \in \mathcal{I} \;:\; \langle f, k \rangle \cap [p] \neq \emptyset \,\}$. We begin by proving that A_p is dense and open in $\mathcal{I}$.

Let $g \in \mathcal{I}^{<\infty}$ be arbitrary and let $\Sigma = \mathrm{dom}(g) \setminus \mathrm{ran}(g)$. Then, by Lemma 4.6, $g \in \mathcal{I}_\Sigma^{<\infty}$. Hence, by Lemma 4.9, there exists $h \in \mathcal{I}_\Sigma^{<\infty}$ extending g such that $\langle f, k \rangle \cap [p] \neq \emptyset$ for all $k \in [h] \cap \mathcal{I}_\Sigma$. Hence if $k \in [h] \cap \mathcal{I}_\Sigma$, then $k \in [h] \cap A_p$. In particular, $[g] \cap A_p \neq \emptyset$ and so A_p is dense in $\mathcal{I}$.

On the other hand, if $k \in A_p$ is arbitrary, then $\langle f, k \rangle \cap [p] \neq \emptyset$. Hence, as in Lemma 2.4, there exists a finite subset Λ of Ω such that for all $h \in [k|_\Lambda] \cap \mathcal{I}$

$$\langle f, h \rangle \cap [p] \neq \emptyset.$$

In other words, $k \in [k|_\Lambda] \cap \mathcal{I} \subseteq A_p$ and so A_p is open in $\mathcal{I}$.

As in the proof of Theorem 1.6, using Lemma 4.1 it is straightforward to show that

$$\bigcap_{p\in\mathcal{P}} A_p = \mathcal{D}_f \cap \mathcal{I}.$$

It follows that $\mathcal{D}_f \cap \mathcal{I}$ is comeagre in $\mathcal{I}$.

It remains to prove that $\mathcal{I}_\infty$ is comeagre in $\mathcal{I}$. Let

$$B_m = \{\, f \in \mathcal{I} \ :\ f \text{ has at least } m \text{ cycles} \,\}.$$

Then, by Lemma 4.10, B_m contains a dense open subset of $\mathcal{I}$. Hence $\mathcal{I}_\infty = \bigcap_{m=1}^{\infty} B_m$ is a comeagre subset of $\mathcal{I}$. Since $\mathcal{D}_f \cap \mathcal{I}$ and $\mathcal{I}_\infty$ are comeagre in $\mathcal{I}$, it follows that $\mathcal{D}_f \cap \mathcal{I}_\infty$ is comeagre in $\mathcal{I}$. □

Proof of Theorem 1.10. For each $n \in \mathbb{N}$ let

$$B_n = \{(f,g) \in \mathrm{Aut}(R_\mathcal{C}) \times \mathrm{Aut}(R_\mathcal{C}) : |\operatorname{fix}(f) \cap \operatorname{fix}(g)| \geq n\}.$$

It is clear that B_n is open in $\mathrm{Aut}(R_\mathcal{C}) \times \mathrm{Aut}(R_\mathcal{C})$. We will show that B_n is dense. Let $(f,g) \in \mathrm{Aut}(R_\mathcal{C})^{<\infty} \times \mathrm{Aut}(R_\mathcal{C})^{<\infty}$. By the Alice's restaurant property we may choose a set $\Gamma \subseteq \Omega$ with more than n elements such that

- $\Gamma \cap [\mathrm{dom}(f) \cup \mathrm{ran}(f) \cup \mathrm{dom}(g) \cup \mathrm{ran}(g)] = \emptyset$; and
- $F\{\alpha,\beta\} = \lambda_1$ for all $\alpha \in \mathrm{dom}(f) \cup \mathrm{ran}(f) \cup \mathrm{dom}(g) \cup \mathrm{ran}(g)$ and $\beta \in \Gamma$.

If f_1, g_1 are extensions of f and g such that $f_1(\alpha) = g_1(\alpha) = \alpha$ for all $\alpha \in \Gamma$, then, by the definition of Γ, $f_1, g_1 \in \mathrm{Aut}(R_\mathcal{C})^{<\infty}$. Moreover, $f_1, g_1 \in B_n$ and so B_n is dense in $\mathrm{Aut}(R_\mathcal{C}) \times \mathrm{Aut}(R_\mathcal{C})$. It follows that $B = \bigcap_{n=1}^{\infty} B_n$ is a comeagre subset of $\mathrm{Aut}(R_\mathcal{C}) \times \mathrm{Aut}(R_\mathcal{C})$.

We next show that each $(f,g) \in B$ generates a nowhere dense subgroup of $\mathrm{Aut}(R_\mathcal{C})$. Let $h \in \mathrm{Aut}(R_\mathcal{C})^{<\infty}$. It suffices to find an extension $k \in \mathrm{Aut}(R_\mathcal{C})^{<\infty}$ of h so that $\langle\, f, g \,\rangle \cap [k] = \emptyset$. As $\operatorname{fix}(f) \cap \operatorname{fix}(g)$ is infinite, we may choose $\alpha \in [\operatorname{fix}(f) \cap \operatorname{fix}(g)] \setminus [\mathrm{dom}(h) \cup \mathrm{ran}(h)]$. By the Alice's restaurant property, we can find $\beta \neq \alpha$ such that k, the extension of h defined by $k(\alpha) = \beta$, is in $\mathrm{Aut}(R_\mathcal{C})^{<\infty}$. This is the desired k as $f(\alpha) = g(\alpha) = \alpha$ and $k(\alpha) \neq \alpha$. □

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