Burnside and his Problem
or
What does a group theorist do anyway?

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20th April 2006

Overview

- What is a group?

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- How do they fit into mathematics?

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- What sort of problems does a group theorist consider?

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- How do they fit into mathematics?
- What sort of problems does a group theorist consider?
- The Burnside Problem

What is a group?

A group is a set G in which we can 'multiply' elements such that

(i) $x(yz) = (xy)z$ for $x, y, z \in G$

(ii) there is an identity element 1 with

$$x1 = 1x = x \quad \text{for } x \in G$$

(iii) every element $x \in G$ has an inverse x^{-1} with

$$xx^{-1} = x^{-1}x = 1.$$

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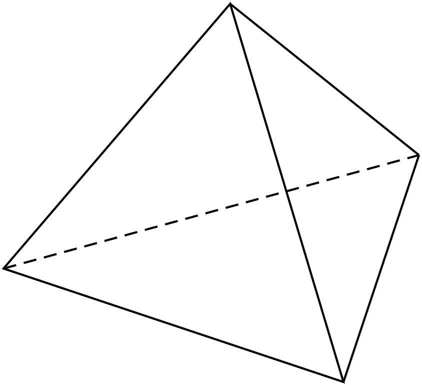
- View $x \in G$ as a transformation of some object X;
- multiplication xy represents composition of transformations;
- 1 is the identity transformation: fix everything in X;
- all transformations are invertible: apply x^{-1} to undo the effect of applying x.

Example: Rotations of the regular tetrahedron



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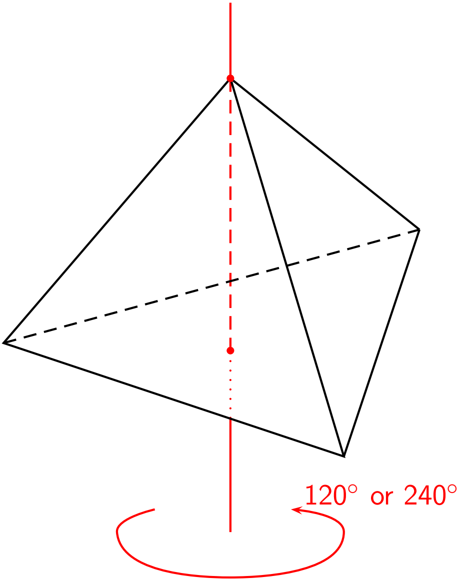

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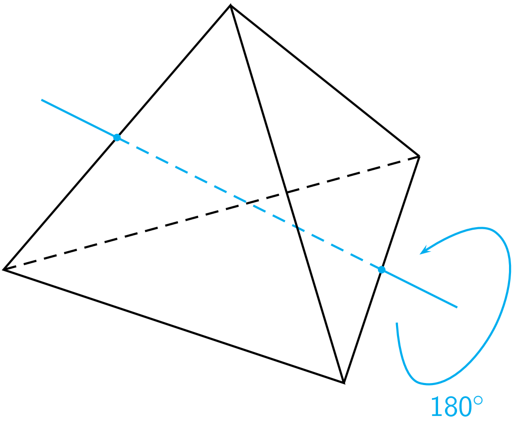

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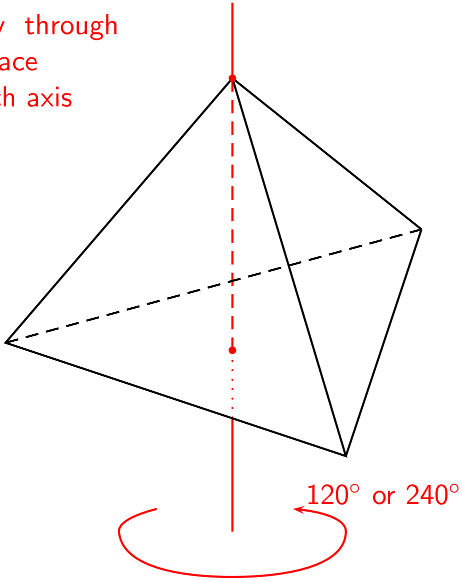
4 axes of symmetry through vertex and opposite face
2 rotations about each axis



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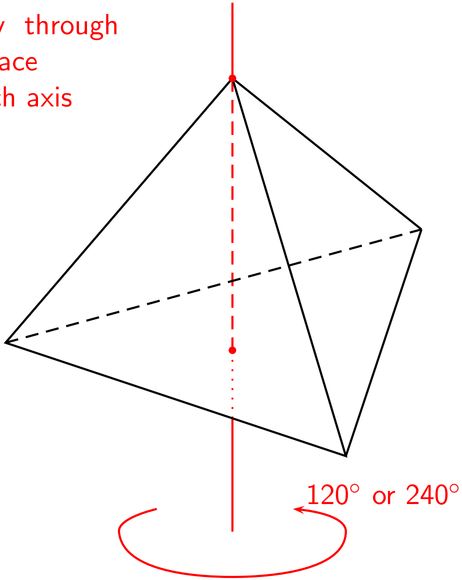
$\Rightarrow$ 8 transformations



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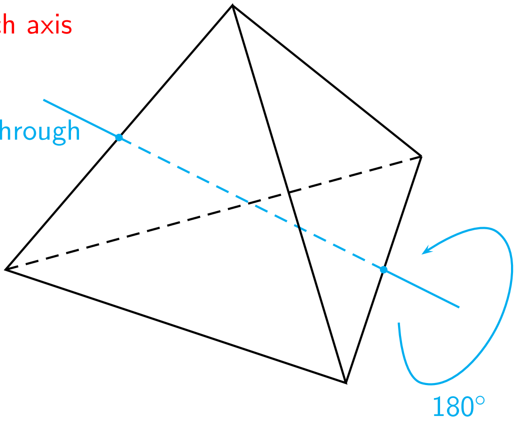
3 axes of symmetry through opposite edges
1 rotation about each axis
$\Rightarrow$ 3 transformations



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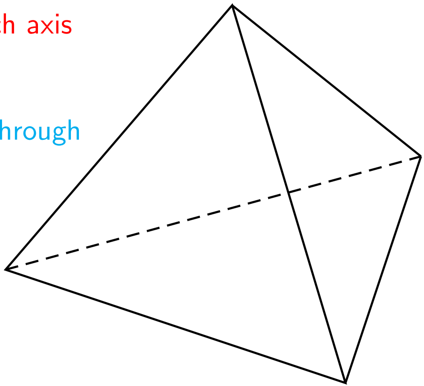

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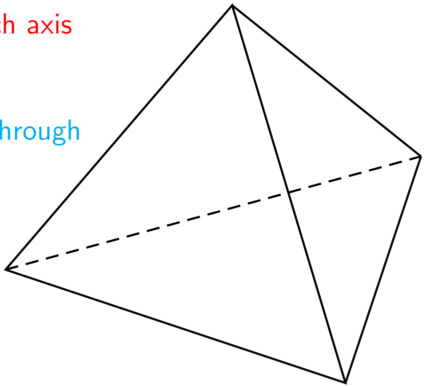

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The symmetry group of the tetrahedron has order 12

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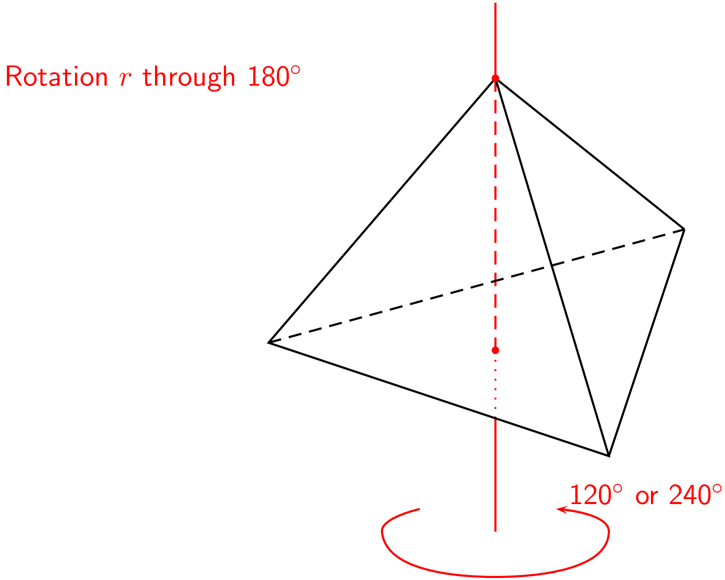

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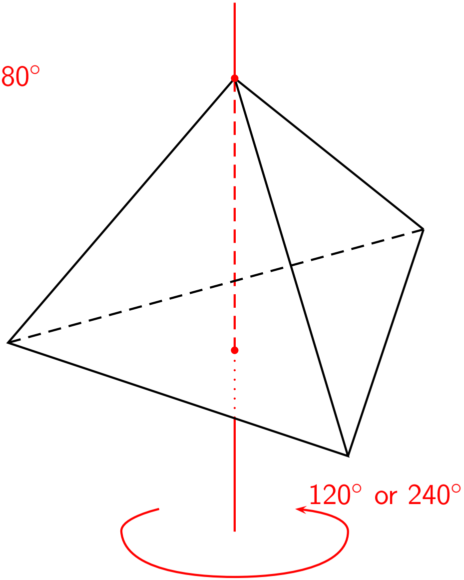
Rotation r through 180°

$r^3 = 1$



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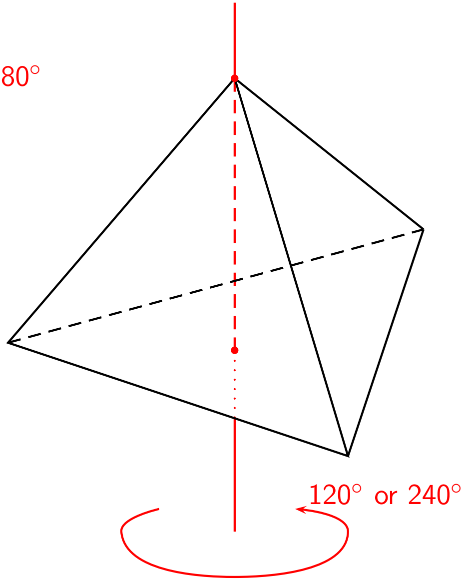

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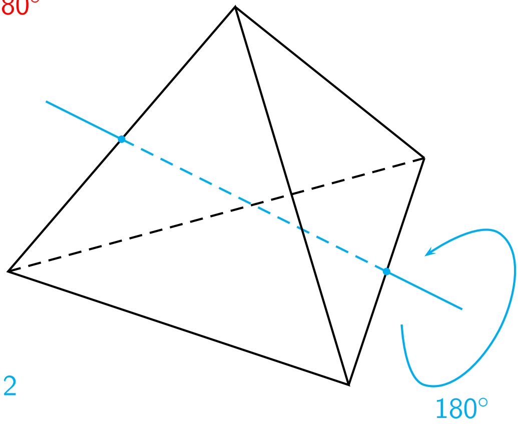

Groups arise as the "symmetries" of "mathematical objects"

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- Galois’s groups consisted of permutations of the roots of polynomial equations;
- By exploiting the group structure, he showed that there is no formula to solve a quintic equation.

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Also, groups are very interesting in their own right!!

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2. checking facts for each group in the list.

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- Groups of prime-power order. Difficult, near impossible.

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Define $f(n)$ to be the number of essentially different groups of order n.

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$$\begin{aligned}
f(1) &= 1\\
f(2) &= 1\\
f(4) &= 2\\
f(8) &= 5\\
f(16) &= 14\\
f(32) &= 51\\
f(64) &= 267\\
f(128) &= 2\,328\\
f(256) &= 56\,092\\
f(512) &= 10\,494\,213\\
f(1024) &= 49\,487\,365\,422
\end{aligned}$$

Enumeration

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Takes up 1000s of pages of journal articles, involved numerous mathematicians.
Classifies these "building blocks" into a number of well-understood families, plus 26 sporadic groups.
The largest of the sporadic simple groups is the Monster of order

$$808\,017\,424\,794\,512\,875\,886\,459\,904\,961\,710\,757\,005\,754\,368\,000\,000\,000$$

(approx. 8×10^{53})

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What is "well-behaved"?

- Collect examples of groups (e.g., symmetry groups of mathematical objects we understand);
- View these as typical and groups sharing similar properties as well-behaved.

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Is it possible for G to be infinite but for every element in G to have finite order?

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General Burnside Problem (GBP)

Let G be a finitely generated group where every element has finite order.
Is G finite?

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Basic Idea

Finite groups are finitely generated and have every element of (bounded) finite order.
All other groups that were familiar at the time do not.
Is there a reason for this?

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So

$$|G| \leqslant n_1 n_2 \ldots n_d.$$

So (GBP) has a positive solution for abelian groups.

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In general, what is the interplay between powers (i.e., elements having finite order) and commutativity?

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(version of the problem for each value of d and n.)

Some results

Example

Burnside Problem has positive solution for exponent 2.

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M. Hall, Jr. (1958):

Burnside Problem has positive solution for exponent 6.

[M. Hall's work described as a "heroic piece of calculation".]

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Adian–Novikov (1968):

There exists an infinite 2-generated group of exponent n for every odd $n \geqslant 4381$.

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Ol'shankii (1982):

Tarski monsters exist: —
For every prime $p > 10^{75}$, there exists an infinite group where every proper subgroup is cyclic of order p.

That's all got a bit technical. . . so what?

That's all got a bit technical... so what?

Things look bad.

Groups aren't at all nicely behaved.

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Perhaps we can rescue things by moving to a class of groups which behave more nicely. . . finite groups?

The Burnside Problem in the world of finite groups

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Restricted Burnside Problem

Is there a largest d-generator group of exponent n?

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Restricted Burnside Problem

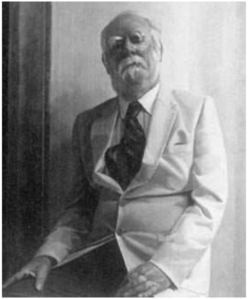
Is there a largest d-generator group of exponent n?

Given the bad news so far, perhaps a surprising answer:

YES!

Three important group theorists

Philip Hall
1904–1982



Graham Higman
1917–

Efim Zelmanov
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It is sufficient to solve the Restricted Burnside Problem for exponent p^k (p prime).

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The Restricted Burnside Problem has a positive solution for all d and n.

1994: Zelmanov awarded the Fields Medal for his solution.

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- Find interesting examples of groups with unusual behaviours;
- Find conditions for when groups are well-behaved
  (i.e., do what we expect).