

# **Groups whose proper quotients are virtually abelian**

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## Setup: Just non- $\mathcal{P}$ groups

**Definition:** Let  $\mathcal{P}$  be a property of groups.  
A group  $G$  is **just non- $\mathcal{P}$**  if

- $G$  does not have  $\mathcal{P}$ ;
- every proper quotient does have  $\mathcal{P}$ .

**Classical Example:**  $\mathcal{P} = \text{“finite”}$

$G$  is **just infinite** if  $G$  is infinite while every proper quotient is finite:

$$1 < N \triangleleft G \implies |G : N| < \infty.$$

### Examples of just infinite groups

- $\mathbb{Z}$
- $D_\infty = \langle x, y \mid y^2 = 1, y^{-1}xy = x^{-1} \rangle$
- any infinite simple group ☹
- many branch groups (e.g., the Grigorchuk groups)

## Why are just non- $\mathcal{P}$ groups interesting?

- They “live on the boundary” of the  $\mathcal{P}$ -groups
- *Every finitely generated infinite group has a just infinite quotient.*

PROOF: Apply Zorn’s Lemma to

$$\mathcal{S} = \{ N \trianglelefteq G \mid |G : N| = \infty \}.$$

The quotient  $G/M$  by a maximal member  $M$  is a just infinite group.

**Idea:** To study infinite groups, first study their just infinite quotients.

**Generalise:** similar result holds whenever f.g.  $\mathcal{P}$ -groups are necessarily *finitely presented*.

- Interesting groups arise!

# Classifying just infinite groups

**McCarthy (1968/1970):** just infinite groups with a non-trivial abelian normal subgroup: —  
 $F(G)$  is a self-centralising abelian normal subgroup.

**Wilson (1971/2000):** just infinite groups with no non-trivial abelian subnormal subgroup.

Two classes:

- finite extensions of direct products of hereditarily just infinite groups;
- branch groups.

[**Hereditarily just infinite:** every subgroup of finite index is just infinite.]

# Generalising: classifying just non- $\mathcal{P}$ groups

McCarthy-style classifications (with non-trivial abelian normal subgroup):

- Newman (1960):  $\mathcal{P} = \text{abelian}$
- Robinson–Wilson (1984):  $\mathcal{P} = \text{polycyclic}$
- Robinson–Zhang (1988):  $\mathcal{P} = \text{finite-by-abelian}$
- Franciosi–de Giovanni (1985):  $\mathcal{P} = \text{Černikov}$
- Franciosi–de Giovanni–Kurdachenko (1996):  
 $\mathcal{P} = \text{FC-groups}$
- etc.
- De Falco (2002):  $\mathcal{P} = \text{nilpotent-by-finite}$

## Wilson-style classifications?

**Fact:** *Every proper quotient of an arbitrary branch group is abelian-by-finite.*

So branch groups are **just non-(abelian-by-finite)** (JNAF-groups).

Hardy, a student of Wilson, in his Ph.D. thesis (2002) produces a Wilson-style classification for JNAF-groups.

Two classes:

- finite extensions of direct products of hereditarily JNAF-groups
- (all) branch groups.

## Obvious question: McCarthy-style classification of JNAF-groups?

**First observation:** Many of De Falco's (JNNF-)groups are *actually JNAF-groups!*

e.g., typical  $G$  is a finite extension of  $A \rtimes X$  where

- $A, X$  are non-trivial torsion-free abelian subgroups
- $A \trianglelefteq G$
- $A$  is a faithful just infinite  $G/A$ -module:  
i.e.,  $C_G(A) = A$ ,  
every proper quotient of  $A$  by a  $G$ -invariant subgroup is finite.

Such a group  $G$  is JNAF. [Proof?]

**Consequence:** We already have a suitable classification of groups which are not nilpotent-by-finite, but every quotient is abelian-by-finite.

# Classifying nilpotent-by-finite JNAF-groups

- Differs from typical McCarthy-style studies:  
 $F(G)$  is non-abelian.
- Need to restrict simple subnormal subgroups instead.

Let  $G$  be a JNAF-group which is nilpotent-by-finite.  
Let  $F = F(G) \leq_f G$  and  $Z = Z(F)$ .

## Case 1: $Z$ has torsion

Let  $X \leq Z$ ,  $X \cong C_p$ . Then

- $X$  is the unique minimal normal subgroup of  $G$
- $G/X$  is abelian-by-finite
- $G$  contains  $K$  nilpotent of class 2, of finite index, with  $K' = X$ .
- $K/Z(K)$  is an elementary abelian  $p$ -group  
(as  $[x^p, y] = [x, y]^p = 1$  for  $x, y \in K$ )
- $K/Z(K)$  must be infinitely generated.

## “Typical” torsion examples

$$K = \langle x_1, y_1, x_2, y_2, \dots, z \rangle$$

subject to

$$x_i^p = y_i^p = z^p = 1, \quad z \in Z(K)$$

$$[x_i, y_j] = \begin{cases} z & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

**Larger centre:** central product amalgamating the cyclic subgroups of order  $p$ :

$$K \circ C_{p^\infty}, \quad K \circ P, \quad K \circ C_{p^\infty} \circ P$$

where  $P$  is a finite  $p$ -group with cyclic centre.

(So class of  $F(G)$  is unbounded.)

### **Insoluble example:**

$S$ : non-abelian simple group such that  $p \mid |M(S)|$ .

$L$ : perfect central extension of  $S$  with  $Z(L) \cong C_p$ .

Central product amalgamating cyclic subgroups of order  $p$ :

$$K \circ L, \quad \text{or} \quad K \circ C_{p^\infty} \circ L, \quad \text{etc.}$$

### **Some action:**

$H$ : finite group with faithful irreducible  $\mathbb{F}_p H$ -module  $V$  of dimension  $d$ .

Use  $K^{(d)}$  as our nilpotent normal subgroup.

View

$$Z(K^{(d)}) \cong V \quad \text{and} \quad K^{(d)}/Z(K^{(d)}) \cong \bigoplus V$$

as  $\mathbb{F}_p H$ -modules and form

$$G = K^{(d)} \rtimes H.$$

Then  $G$  is JNAF. (Any non-trivial normal subgroup of  $G$  meets  $K^{(d)}$  and hence contains the whole of  $Z(K^{(d)})$  by irreducibility of  $V$ .)

## Classification (torsion case)

### Theorem (MRQ 2005)

*A nilpotent-by-finite JNAF-group  $G$  where  $Z = Z(\mathbb{F}(G))$  possesses torsion has the following structure:*

- (i)  *$Z$  is a  $p$ -primary group (for some prime  $p$ ),*
- (ii) *there exists  $K \trianglelefteq_f G$ , nilpotent of class two, such that*
  - *$K/Z(K)$  is not finitely generated,*
  - *$K'$  is the unique minimal  $G/\mathbb{F}(G)$ -submodule of  $Z$ ,*
- (iii)  *$G$  possesses no non-abelian simple subnormal subgroup.*

## Case 2: $Z$ torsion-free

### Theorem (MRQ 2005)

A nilpotent-by-finite JNAF-group  $G$  where  $Z = Z(\mathbb{F}(G))$  is torsion-free has the following structure:

- (i) there exists  $K \trianglelefteq_f G$ , nilpotent of class two, such that
  - $K'$  is free abelian,
  - $C_K(K/(K')^m) \leq_f K$  for every  $m \in \mathbb{N}$ ,
- (ii) every non-zero  $G/\mathbb{F}(G)$ -submodule of  $Z$  contains a submodule of finite index in  $K'$   
(so  $K'$  is rationally irreducible)
- (iii)  $G$  possesses no non-abelian simple subnormal subgroup.

(Most conditions are direct analogues of the torsion-case.)

## Centraliser condition in torsion-free case

The condition

$$|K : C_K(K/(K')^m)| < \infty \quad \text{for } m \in \mathbb{N}$$

could be viewed as saying  $K/Z(K)$  must not be *too large*:

- if  $K/Z(K)$  is finitely generated, the condition holds;
- if the condition holds, then  $|K/Z(K)| \leq 2^{\aleph_0}$ .

(Contrary to the torsion case, when  $K/Z(K)$  is required to be *large*, namely not finitely generated.)

$$K/Z(K) = 2^{\aleph_0}$$

Let  $o(x_i) = o(y_j) = o(z) = \infty$ ,

$$X = \bigoplus_{i \in \mathbb{N}} \langle x_i \rangle, \quad Y = \prod_{i \in \mathbb{N}} \langle y_i \rangle,$$

$$K = \langle X, Y, z \rangle$$

subject to  $z \in Z(K)$  and

$$[x_i, y_j] = \begin{cases} z^{m_i} & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

where

$$m_{i+1} = p_1^i p_2^i \cdots p_i^i$$

(for some enumeration  $p_1, p_2, \dots$  of primes).

Then  $C_K(K/(K')^m)$  contains almost all  $x_i$  and  $y_j$ ,  
so

$$|K : C_K(K/(K')^m)| < \infty \quad \text{for } m \in \mathbb{N}.$$