Some groups which just aren't quite finite

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Motivation

Typical question in infinite group theory

Let $\mathscr{P}$ be a property of groups.
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- Is G finite?
- Is G soluble-by-finite?
- etc.

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Proposition

Every finitely generated infinite group has a just infinite quotient.

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- Classify the just infinite groups.

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- Classify the just infinite groups.
- Obtain a contradiction by showing none of the just infinite groups do satisfy $\mathscr{P}$.

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Just infinite groups having no non-trivial abelian subnormal subgroup: either

- a finite extension of a direct product of copies of a hereditarily just infinite group, or
- a branch group.

G is hereditarily just infinite if every subgroup of finite index is just infinite.

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Hardy 2002 (Ph.D. Birmingham)

Produces Wilson-style classification for just non-(abelian-by-finite) groups with no abelian-by-finite subnormal subgroups.

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- etc.

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Typically focus on

$$\mathrm{F}(G) = \langle\, N \trianglelefteq G \mid N \text{ nilpotent} \,\rangle,$$

viewed as a module.

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(iii) $AX = A \rtimes X \leqslant_f G$.

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If α is a non-degenerate bilinear form on a vector space V of dimension n, then a totally isotropic subspace U $(\alpha \equiv 0$ on $U)$ has $\operatorname{codim} U \geqslant n/2$.

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Fact

If G is soluble, then $\mathsf{C}_G(\mathsf{F}(G)) \leqslant \mathsf{F}(G)$.

Generalised Fitting subgroup

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Fact

In finite group theory, $\mathsf{C}_G(\mathsf{F}^*(G)) \leqslant \mathsf{F}^*(G)$.
Also true if $\mathsf{F}(G) \leqslant_f G$.

Theorem (MRQ, 2005)

Classification of nilpotent-by-finite JNAF-groups with Z torsion:

(i) *Z is a p-primary group (some prime p),*

(ii) *there exists $K \trianglelefteq_f G$, nilpotent of class two, such that*

- *$K/\mathsf{Z}(K)$ is not f.g.,*
- *K' is the unique minimal $G/\mathsf{F}(G)$-submodule of Z,*

(iii) *every component of G has non-trivial centre.*

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Classification of nilpotent-by-finite JNAF-groups with Z torsion-free:

(i) *there exists $K \trianglelefteq_f G$, nilpotent of class two, such that*
  - K' *is free abelian,*
  - $\mathsf{C}_K(K/(K')^m) \leqslant_f K$ *for all m,*

(ii) *every non-trivial $G/\mathsf{F}(G)$-submodule of Z contains a submodule of finite index in K',*

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Last condition of (i) implies $|K/\mathsf{Z}(K)| \leqslant 2^{\aleph_0}$.
The condition holds if $K/\mathsf{Z}(K)$ is f.g.