Some groups which just aren't quite finite

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Motivation

Typical question in infinite group theory

Let $\mathscr{P}$ be a property of groups.
If G satisfies $\mathscr{P}$, is it necessarily well-behaved?

- Is G finite?
- Is G soluble-by-finite?
- etc.

Just infinite groups

Definition
G is just infinite if
(i) G is infinite;
(ii) if $\mathbf{1} < N \trianglelefteq G$, then $|G : N| < \infty$.

Examples

- $\mathbb{Z}$
- $D_\infty = \langle\, x, y \mid y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$
- any infinite simple group

Proposition
Every finitely generated infinite group has a just infinite quotient.

Strategy for proving $\mathscr{P}$ implies finiteness

- Let $\mathscr{P}$ be a property inherited by quotients.
- Let G be a f.g. infinite group satisfying $\mathscr{P}$.
- Then G has a just infinite quotient also satisfying $\mathscr{P}$.
- Classify the just infinite groups.
- Obtain a contradiction by showing none of the just infinite groups do satisfy $\mathscr{P}$.

Classifying just infinite groups

McCarthy 1968

Classification result for just infinite groups with non-trivial abelian normal subgroup.

Wilson 1971/2000

Just infinite groups having no non-trivial abelian subnormal subgroup: either

- a finite extension of a direct product of copies of a hereditarily just infinite group, or
- a branch group.

G is hereditarily just infinite if every subgroup of finite index is just infinite.

Generalisations I

Branch groups are just non-(abelian-by-finite) (JNAF) groups. (Every proper quotient is abelian-by-finite.)

Question (Grigorchuk)

Is a f.g. branch group always just infinite?

Hardy 2002 (Ph.D. Birmingham)

Produces Wilson-style classification for just non-(abelian-by-finite) groups with no abelian-by-finite subnormal subgroups.

Generalisations II

Lots of work on just non-$\mathscr{P}$ groups with abelian normal subgroup:

- Newman (1960): $\mathscr{P}$ = abelian
- Robinson–Wilson (1984): $\mathscr{P}$ = polycyclic
- Robinson–Zhang (1988): $\mathscr{P}$ = finite-by-abelian
- Franciosi–de Giovanni (1985): $\mathscr{P}$ = Černikov
- Franciosi–de Giovanni–Kurdachenko (1996): $\mathscr{P}$ = FC-group
- etc.
- de Falco (2002): $\mathscr{P}$ = nilpotent-by-finite.

Typically focus on

$$\mathsf{F}(G) = \langle\, N \trianglelefteq G \mid N \text{ nilpotent} \,\rangle,$$

viewed as a module.

Theorem (de Falco, 2002)

Suppose $\mathsf{F}(G)$ *has finite rank. Classification of JNNF-groups with* $\mathsf{F}(G)$ *non-trivial torsion-free:*
G *has non-trivial torsion-free abelian subgroups* A, X *with*

(i) $A \trianglelefteq G$,

(ii) A *is a faithful just infinite* G/A*-module,*

(iii) $AX = A \rtimes X \leqslant_f G$.

Classifying nilpotent-by-finite JNAF-groups

Let G be a JNAF-group which is nilpotent-by-finite.

Lemma
If $\mathbf{1} < M, N \trianglelefteq G$, *then* $M \cap N \neq \mathbf{1}$.

Fact
If α is a non-degenerate bilinear form on a vector space V of dimension n, then a totally isotropic subspace U $(\alpha \equiv 0$ on $U)$ has $\operatorname{codim} U \geqslant n/2$.

Fact
If G is soluble, then $\mathsf{C}_G(\mathsf{F}(G)) \leqslant \mathsf{F}(G)$.

Generalised Fitting subgroup

A component L of G is a *subnormal quasisimple* subgroup:
$L \trianglelefteq \cdots \trianglelefteq G$, $L' = L$ and $L/\mathsf{Z}(L)$ is simple.

Generalised Fitting subgroup:

$$\mathsf{F}^*(G) = \mathsf{F}(G) \cdot \langle\, L \mid L \in \mathsf{Comp}(G)\,\rangle.$$

Fact
In finite group theory, $\mathsf{C}_G(\mathsf{F}^*(G)) \leqslant \mathsf{F}^*(G)$.
Also true if $\mathsf{F}(G) \leqslant_f G$.

Theorem (MRQ, 2005)

Classification of nilpotent-by-finite JNAF-groups with Z torsion:

(i) *Z is a p-primary group (some prime p),*

(ii) *there exists $K \trianglelefteq_f G$, nilpotent of class two, such that*
- *$K/\mathsf{Z}(K)$ is not f.g.,*
- *K' is the unique minimal $G/\mathsf{F}(G)$-submodule of Z,*

(iii) *every component of G has non-trivial centre.*

Theorem (MRQ, 2005)

Classification of nilpotent-by-finite JNAF-groups with Z torsion-free:

(i) *there exists $K \trianglelefteq_f G$, nilpotent of class two, such that*
- *K' is free abelian,*
- *$\mathsf{C}_K(K/(K')^m) \leqslant_f K$ for all m,*

(ii) *every non-trivial $G/\mathsf{F}(G)$-submodule of Z contains a submodule of finite index in K',*

(iii) *G has no components.*

Last condition of (i) implies $|K/\mathsf{Z}(K)| \leqslant 2^{\aleph_0}$.
The condition holds if $K/\mathsf{Z}(K)$ is f.g.