Some groups which just aren't quite finite

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Motivation

Typical question in infinite group theory

Let $\mathscr{P}$ be a property of groups.
If G satisfies $\mathscr{P}$, is it necessarily well-behaved?

- Is G finite?
- Is G soluble-by-finite?
- etc.

Just infinite groups

Definition

G is just infinite if

(i) G is infinite;

(ii) every proper quotient is finite:

$$\mathbf{1} < N \trianglelefteq G \implies |G : N| < \infty.$$

Examples

- $\mathbb{Z}$
- $D_\infty = \langle\, x, y \mid y^2 = 1,\ y^{-1}xy = x^{-1}\,\rangle$
- any infinite simple group ☹
- many branch groups (e.g., the Grigorchuk groups)

Proposition

Every finitely generated infinite group has a just infinite quotient.

Strategy (?) for proving $\mathscr{P}$ implies finiteness

- Let $\mathscr{P}$ be a property inherited by quotients.
- Let G be a f.g. infinite group satisfying $\mathscr{P}$.
- Then G has a just infinite quotient also satisfying $\mathscr{P}$.
- Classify the just infinite groups.
- Obtain a contradiction by showing none of the just infinite groups do satisfy $\mathscr{P}$.

Classifying just infinite groups

McCarthy 1968/1970

Classification result for just infinite groups with non-trivial abelian normal subgroup: —
$\mathrm{F}(G)$ is a self-centralising abelian normal subgroup.

Wilson 1971/2000

Just infinite groups having no non-trivial abelian subnormal subgroup.
Two classes:

- finite extensions of direct products of hereditarily just infinite groups;
- branch groups.

[Hereditarily just infinite: *every* subgroup of finite index is just infinite.]

Generalisation

Definition

Let $\mathscr{P}$ be a property of groups.
A group G is just non-$\mathscr{P}$ if

- G does not have property $\mathscr{P}$;
- every proper quotient does have property $\mathscr{P}$.

Why are these interesting?

- They "live on the boundary" of the $\mathscr{P}$-groups.
- Just non-$\mathscr{P}$ quotients exist (provided f.g. $\mathscr{P}$-groups are *finitely presented*).
- Branch groups are always just non-(abelian-by-finite). [JNAF-group]
- Interesting groups arise!

Classifying just non-$\mathscr{P}$ groups

McCarthy-style classifications (with non-trivial abelian normal subgroup):

- Newman (1960): $\mathscr{P}$ = abelian
- Robinson–Wilson (1984): $\mathscr{P}$ = polycyclic
- Robinson–Zhang (1988): $\mathscr{P}$ = finite-by-abelian
- Franciosi–de Giovanni (1985): $\mathscr{P}$ = Černikov
- Franciosi–de Giovanni–Kurdachenko (1996): $\mathscr{P}$ = FC-group
- etc.
- De Falco (2002): $\mathscr{P}$ = nilpotent-by-finite.

Wilson-style classifications?

Some branch groups occur in Wilson's classification, while *all* branch groups are JNAF-groups.

Hardy, a student of Wilson, in his Ph.D. thesis (2002) produces a Wilson-style classification for JNAF-groups.
Two classes:

- finite extensions of direct products of hereditarily JNAF-groups;
- (all) branch groups.

Obvious Question: McCarthy-style classification of JNAF-groups?

First observation

Many of De Falco's (JNNF-)groups are *actually JNAF-groups*!

e.g., typical G is a finite extension of $A \rtimes X$ where

- A, X are non-trivial torsion-free abelian subgroups
- $A \trianglelefteq G$
- A is a faithful just infinite G/A-module:
  i.e., $\mathsf{C}_G(A) = A$,
  every proper quotient of A by a G-invariant subgroup is finite.

Consequence

We already have a suitable classification of those groups which are not nilpotent-by-finite but every quotient is abelian-by-finite.

Classifying nilpotent-by-finite JNAF-groups

- Differs from typical McCarthy-style studies:
  $\mathsf{F}(G)$ is non-abelian.
- Need to restrict simple subnormal subgroups instead.

Let G be a JNAF-group which is nilpotent-by-finite.
$F = \mathsf{F}(G) \leqslant_f G, \quad Z = \mathsf{Z}(F)$.

Lemma

If $\mathbf{1} < M, N \trianglelefteq G$, *then* $M \cap N \neq \mathbf{1}$.

Case 1: Z possesses torsion

Let $X \leqslant Z$, $X \cong C_p$. Then

- X is the unique minimal normal subgroup of G
- G/X is abelian-by-finite
- G contains K nilpotent of class 2, of finite index, with $K' = X$
- $K/\mathsf{Z}(K)$ is an elementary abelian p-group
- $K/\mathsf{Z}(K)$ must be infinitely generated.

Fact

If α is a non-degenerate bilinear form on a vector space V of dimension n, then a totally isotropic subspace U ($\alpha \equiv 0$ on U) has $\operatorname{codim} U \geqslant n/2$.

Classification (torsion case)

Theorem (MRQ, 2005)

A nilpotent-by-finite JNAF-group G where $Z = \mathsf{Z}(\mathsf{F}(G))$ has torsion has the following structure:

(i) *Z is a p-primary group (some prime p),*

(ii) *there exists $K \trianglelefteq_f G$, nilpotent of class two, such that*
   - *$K/\mathsf{Z}(K)$ is not f.g.,*
   - *K' is the unique minimal $G/\mathsf{F}(G)$-submodule of Z,*

(iii) *G possesses no non-abelian simple subnormal subgroup.*

Case 2: Z torsion-free

Theorem (MRQ, 2005)

A nilpotent-by-finite JNAF-group G where $Z = \mathrm{Z}(\mathrm{F}(G))$ is torsion-free has the following structure:

(i) *there exists $K \trianglelefteq_f G$, nilpotent of class two, such that*
- *K' is free abelian*
- *$\mathrm{C}_K(K/(K')^m) \leqslant_f K$ for all m*

(ii) *every non-zero $G/\mathrm{F}(G)$-submodule of Z contains a submodule of finite index in K'*
(so K' is rationally irreducible)

(iii) *G possesses no non-abelian simple subnormal subgroup.*

Most conditions are direct analogues of the torsion-case.

Centraliser condition

The condition

$$|K : \mathrm{C}_K(K/(K')^m)| < \infty \qquad \text{for } m \in \mathbb{N}$$

can be viewed as saying $K/\mathrm{Z}(K)$ must not be too large:

- if $K/\mathrm{Z}(K)$ is f.g., the condition holds;
- if the condition holds, then $|K/\mathrm{Z}(K)| \leqslant 2^{\aleph_0}$.

(cf. torsion case, when $K/\mathrm{Z}(K)$ is required to be large, namely not f.g.)