

On the Growth of Generating Sets for Direct Powers of 2-element Algebras

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For an algebra A let $d(A)$ denote the smallest number of generators needed to generate A , and let

$$\mathbf{d}(A) = (d(A), d(A^2), d(A^3), \dots),$$

where A^n is the n th direct power of A . In [6] Wiegold shows that $\mathbf{d}(G)$ for a non-trivial finite group G grows logarithmically if G is perfect (equal to its derived subgroup), and linearly otherwise. In another paper [7] Wiegold shows that $\mathbf{d}(S)$ for a finite non-group semigroup grows linearly if S is a monoid (has an identity element), and exponentially otherwise. Wiegold's results for finite groups have been generalised in [5] to finite structures in other 'classical' varieties (rings, modules, [associative] algebras, Lie algebras). For instance, it is shown there that for a non-trivial finite ring R , $\mathbf{d}(R)$ grows logarithmically if R has identity, and grows linearly otherwise.

In all the investigations so far no finite non-trivial algebra has been encountered the \mathbf{d} -sequence of which would grow at a rate different from logarithmic, linear or exponential. The purpose of this note is to record the following observation:

Theorem. *The sequence $\mathbf{d}(A)$, where A is a 2-element algebra, grows logarithmically, linearly or exponentially.*

The key for proving this is, of course, Post's classification [4] of all algebras on the set $\{0, 1\}$. Also crucial is the following observation from [5, Section 2]: for a finite algebra A the sequence $\mathbf{d}(A)$ grows logarithmically (respectively, linearly or exponentially) if and only if the \mathbf{d} -sequence of the algebra obtained by adding all the constants to A grows logarithmically (respectively, linearly or exponentially).

Thus we only ought to consider the algebras from Post's classification (term-equivalent to algebras) that contain both constants 0 and 1. Below we list them all; in doing so we omit the carrier set which is always $\{0, 1\}$, and also we omit the constants 0 and 1 from the list of operations.

- The algebra with no non-constant operations. The \mathbf{d} -sequence of this algebra is obviously $(2^n - 2)$, and grows exponentially.

Presented by ...

Received ...; accepted in final form ...

2000 *Mathematics Subject Classification*: Primary: 08A10.

Key words and phrases: generators, direct powers.

- $O_9 = (\neg)$, where \neg is the unary operation of negation. Here clearly $d(O_9^n) = 2^{n-1} - 1$, and the \mathbf{d} -sequence grows exponentially.
- $S_6 = (\vee)$, where \vee is the binary ‘or’ operation. Clearly, S_6 is the two element semilattice, i.e. a non-group monoid, and the results of [7] apply to show that $\mathbf{d}(S_6)$ is linear.
- $A_1 = (\vee, \wedge)$, where \wedge is the binary ‘and’ operation. This is the two-element lattice. If we added the unary operation \neg to the signature of A_1 , we would obtain an algebra polynomially equivalent to the algebra C_1 below, which is shown to have a logarithmic \mathbf{d} -sequence. Now, if X is any set that generates C_1^n , then the set $X \cup \neg X$ generates A_1^n , because of the de Morgan laws. It therefore follows that $\mathbf{d}(A_1)$ also grows logarithmically.
- $L_1 = (+)$, where $+$ is the usual addition modulo 2. This is the cyclic group of order 2, it is clearly polynomially equivalent to L_1 considered as a two-element vector space over the two-element field, and hence has a linear \mathbf{d} -sequence.
- $C_1 = (+, \wedge)$. This is the boolean ring on two elements. It clearly has an identity and so the results of [5] apply to show that $\mathbf{d}(C_1)$ is logarithmic.

In closing we mention that the reasoning above for C_1 is linked with functional completeness, as explained in [5]. Also, the result above concerning L_1 is a very special case of a more general theory applying to lattices and wider, which will be treated separately in a forthcoming publication [3]. We also remark that Chen [1, 2] works with \mathbf{d} -sequences without calling them such, and apparently without being aware of Wiegold’s pioneering work. Our observation in this note refines [1, Proposition 5], [2, Proposition 9], where it is shown that the growth of the \mathbf{d} -sequence of a two element algebra is either bounded above by a polynomial or is exponential.

Acknowledgements. This paper arose from the work of the St Andrews Research Summer School 2009, of which the first three authors were participants. The first author’s attendance at the Summer School was financially supported by EPSRC, second by the School of Mathematics and Statistics, and third by the Nuffield Foundation. We would like to thank all the other participants at the Summer School, and especially Samuel Baynes, for useful conversations regarding this project. The fourth and fifth authors would also like to thank Arthur Geddes for their conversations with him regarding various aspects of the growth of \mathbf{d} -sequences in the universal algebraic context.

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