Introduction to Profinite Groups

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Let G be a f.g. residually finite group and define

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Then G has PSG ($s_n(G) \leqslant n^c$ for some c)
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Inverse Systems

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A *directed set* is a partially ordered set I such that if $i_1, i_2 \in I$ there is some $j \in I$ such that $i_1 \leqslant j$ and $i_2 \leqslant j$.

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$$\phi_{ij}\phi_{jk} = \phi_{ik} \qquad \text{when } i \geqslant j \geqslant k.$$

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Note $\varprojlim G_i \subseteq \prod_{i\in I} G_i$.

Lemma

$\varprojlim G_i$ is a subgroup of $\prod_{i\in I} G_i$.

Universal Property of Inverse Limits

Theorem

Let $\{G_i, \phi_{ij}\}_I$ *be an inverse system of groups,* $G = \varprojlim G_i$ *and* $\rho_i \colon G \to G_i$ *the projection to the* i*th component.*

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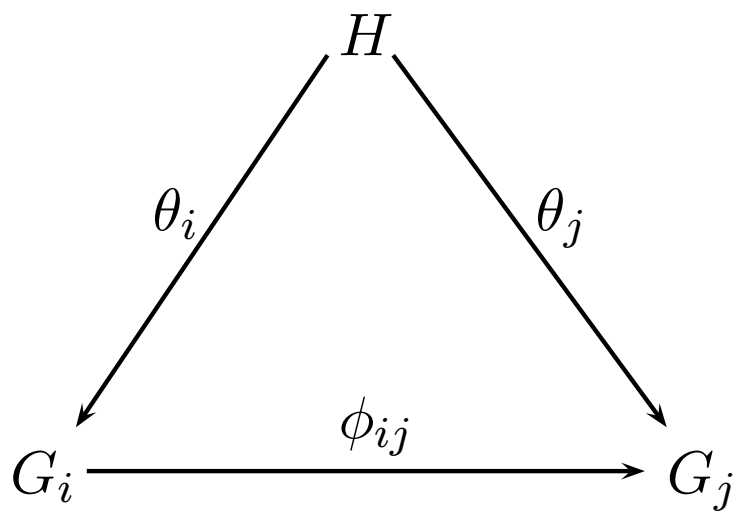

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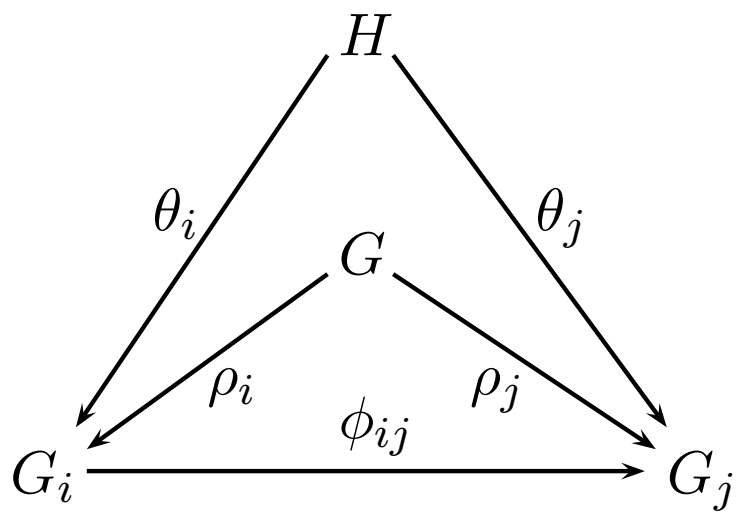

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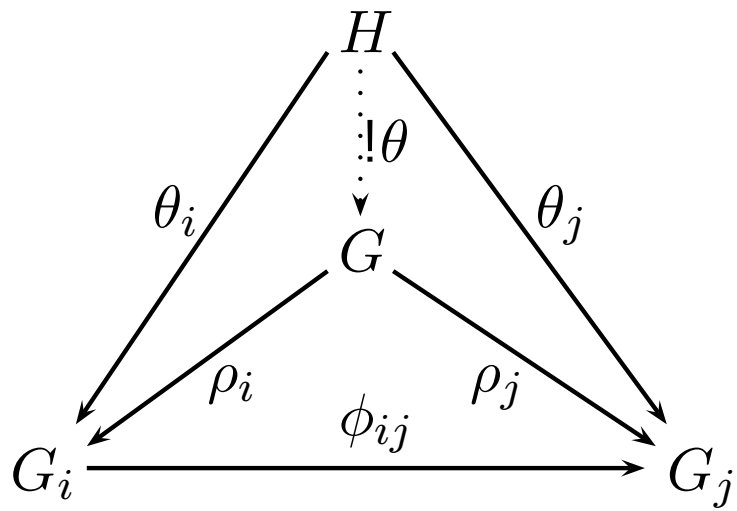

Some topology

Definition

A *topological space* X is a set together with a family $\mathscr{T}$ of *open sets* such that

(T1) $\varnothing, X \in \mathscr{T}$;

(T2) if $U_\lambda \in \mathscr{T}$ for $\lambda \in \Lambda$, then $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathscr{T}$;

(T3) if $U_1, U_2, \ldots, U_n \in \mathscr{T}$, then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathscr{T}$.

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$$X = U_{\lambda_1} \cup U_{\lambda_2} \cup \cdots \cup U_{\lambda_k}.$$

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Example

If X is a finite set, the *discrete topology* is that where *all* subsets of X are decreed to be open.
With this topology, X is compact and Hausdorff.

Back to inverse limits

Suppose $\{G_i, \phi_{ij}\}_I$ is an inverse system of *finite* groups.
View each G_i as a topological space via the discrete topology.

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Proposition

The inverse limit $\varprojlim G_i$ of an inverse system of finite groups is a compact Hausdorff topological group.

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Lemma

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Theorem

Let G satisfy the second part of the Definition. Then

$$G \cong \varprojlim (G/N)_{N \trianglelefteqslant_O G}.$$

Finite images of abstract groups

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Typically $\operatorname{im}\theta < G$, but it *is* dense.

Profinite Completions

Definition

If Γ is an arbitrary group, the inverse limit

$$\hat{\Gamma} = \varprojlim (\Gamma/N)_{N \trianglelefteqslant_f \Gamma}$$

is called the *profinite completion* of Γ.

Theorem

If Γ is residually finite, then it embeds as a dense subgroup of its profinite completion $\hat{\Gamma}$.

A final number theoretic example

Let $\bar{\mathbb{Q}}$ be the *algebraic closure* of $\mathbb{Q}$:

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Further reading

- J. D. Dixon, M. P. F. du Sautoy, A. Mann & D. Segal, *Analytic pro-p groups, 2nd edition*, Cambridge studies in advanced math. **61** (CUP, 1999)
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