

Introduction to Profinite Groups

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Motivation

Frequent theme in infinite group theory

How do the finite images of a group affect its structure?

Theorem (Hirsch, 1946)

Let G be polycyclic-by-finite. If every finite quotient of G is nilpotent, then G is nilpotent.

Theorem (Grunewald–Pickel–Segal, 1980)

Let G be polycyclic-by-finite. Then there are finitely many isomorphism classes of polycyclic-by-finite groups having the same finite quotients as G .

Motivation II

Theorem (Zelmanov, RBP)

For every pair of integers d and n , there is a largest finite d -generator group of exponent n .

Positive Solution of the Restricted Burnside Problem:

A f.g. profinite group of finite exponent is finite.

Theorem (Lubotzky–Mann–Segal, PSG-Theorem, 1993)

Let G be a f.g. residually finite group and define

$$s_n(G) = \#\{H \leq G \mid |G : H| \leq n\}.$$

Then G has **PSG** ($s_n(G) \leq n^c$ for some c)
if and only if G is virtually soluble of finite rank.

Inverse Systems

Definition

A **directed set** is a partially ordered set I such that if $i_1, i_2 \in I$ there is some $j \in I$ such that $i_1 \leq j$ and $i_2 \leq j$.

Definition

An **inverse system**

$$\{G_i, \phi_{ij} \mid i, j \in I, i \geq j\}$$

of groups indexed by a directed set I consists of

- ▶ a group G_i for each $i \in I$
- ▶ a homomorphism $\phi_{ij}: G_i \rightarrow G_j$ whenever $i \geq j$ in I such that

$$\phi_{ij}\phi_{jk} = \phi_{ik} \quad \text{when } i \geq j \geq k.$$

Inverse Limits

Definition

The *inverse limit* of the inverse system $\{G_i, \phi_{ij}\}_I$ is

$$\varprojlim G_i = \{ (g_i)_{i \in I} \mid g_i \in G_i \text{ for all } i \\ \text{and } g_i \phi_{ij} = g_j \text{ whenever } i \geq j \}.$$

Note $\varprojlim G_i \subseteq \prod_{i \in I} G_i$.

Lemma

$\varprojlim G_i$ is a subgroup of $\prod_{i \in I} G_i$.

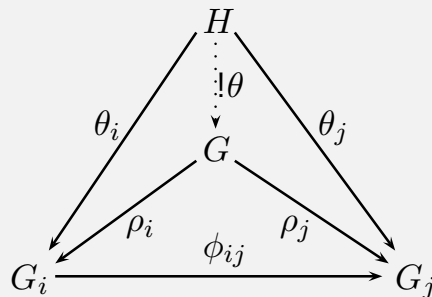
Universal Property of Inverse Limits

Theorem

Let $\{G_i, \phi_{ij}\}_I$ be an inverse system of groups, $G = \varprojlim G_i$ and $\rho_i: G \rightarrow G_i$ the projection to the i th component.

Then (G, ρ_i) is the *unique* solution of the following universal problem:

Given a group H and homomorphisms $\theta_i: H \rightarrow G_i$ such that $\theta_i \phi_{ij} = \theta_j$ whenever $i \geq j$, there is a unique homomorphism $\theta: H \rightarrow G$ such that $\theta_i = \theta \rho_i$ for all i .



Some topology

Definition

A *topological space* X is a set together with a family \mathcal{T} of *open sets* such that

(T1) $\emptyset, X \in \mathcal{T}$;

(T2) if $U_\lambda \in \mathcal{T}$ for $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$;

(T3) if $U_1, U_2, \dots, U_n \in \mathcal{T}$, then $U_1 \cap U_2 \cap \dots \cap U_n \in \mathcal{T}$.

Definition

A topological space X is *compact* if whenever

$$X = \bigcup_{\lambda \in \Lambda} U_\lambda \quad (U_\lambda \text{ open}),$$

there exist $\lambda_1, \lambda_2, \dots, \lambda_k$ such that

$$X = U_{\lambda_1} \cup U_{\lambda_2} \cup \dots \cup U_{\lambda_k}.$$

Definition

A topological space X is *Hausdorff* if whenever $x, y \in X$ ($x \neq y$) there are *disjoint* open sets U, V such that $x \in U$ and $y \in V$.

Example

If X is a finite set, the *discrete topology* is that where *all* subsets of X are decreed to be open.

With this topology, X is compact and Hausdorff.

Back to inverse limits

Suppose $\{G_i, \phi_{ij}\}_I$ is an inverse system of *finite* groups.
View each G_i as a topological space via the discrete topology.

Let $C = \prod_{i \in I} G_i$ have the *product topology*.

Tychonoff's Theorem $\Rightarrow C$ is compact.

Easy to see C is Hausdorff.

$\varprojlim G_i = \{ (g_i) \mid g_i \phi_{ij} = g_j \ \forall i \geq j \}$ is closed in C .

Proposition

The inverse limit $\varprojlim G_i$ of an inverse system of finite groups is a compact Hausdorff topological group.

Profinite Groups

Definition

A *profinite group* is

- ▶ an inverse limit of finite groups, or equivalently
- ▶ a compact Hausdorff topological group such that every open set containing 1 (*neighbourhood*) necessarily contains an open subgroup.

Lemma

If G is a compact topological group, then every open subgroup has finite index.

Theorem

Let G satisfy the second part of the Definition. Then

$$G \cong \varprojlim (G/N)_{N \triangleleft_o G}.$$

Finite images of abstract groups

Let Γ be an (infinite abstract) group.

Then $\{\Gamma/N \mid N \trianglelefteq_f \Gamma\}$ forms an inverse system.

Form $G = \varprojlim (\Gamma/N)_{N \trianglelefteq_f \Gamma}$.

Define

$$\begin{aligned}\theta: \Gamma &\rightarrow G \\ x &\mapsto (Nx)_{N \trianglelefteq_f \Gamma}\end{aligned}$$

$$\ker \theta = \bigcap_{N \trianglelefteq_f \Gamma} N \quad (\textit{finite residual of } \Gamma).$$

Typically $\text{im } \theta < G$, but it *is* dense.

Profinite Completions

Definition

If Γ is an arbitrary group, the inverse limit

$$\hat{\Gamma} = \varprojlim (\Gamma/N)_{N \trianglelefteq_f \Gamma}$$

is called the *profinite completion* of Γ .

Theorem

If Γ is residually finite, then it embeds as a dense subgroup of its profinite completion $\hat{\Gamma}$.

A final number theoretic example

Let $\bar{\mathbb{Q}}$ be the *algebraic closure* of \mathbb{Q} :

$$\begin{aligned}\bar{\mathbb{Q}} &= \{ \alpha \in \mathbb{C} \mid p(\alpha) = 0 \text{ for some } p \in \mathbb{Q}[X] \} \\ &= \bigcup_{p \in \mathbb{Q}[X]} K_p\end{aligned}$$

where K_p is the splitting field of the polynomial p .

These splitting fields K form a directed set.

An automorphism ϕ of $\bar{\mathbb{Q}}$ is determined by its effect on all the splitting fields.

Thus


$$\phi \mapsto (\phi|_K)$$

is an isomorphism

$$\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cong \varprojlim \text{Gal}(K/\mathbb{Q}),$$

so the absolute Galois group is a profinite group.

Further reading

-  J. D. Dixon, M. P. F. du Sautoy, A. Mann & D. Segal, *Analytic pro- p groups, 2nd edition*, Cambridge studies in advanced math. **61** (CUP, 1999)
-  Luis Ribes & Pavel Zalesskii, *Profinite groups*, Ergebnisse der Math. **40** (Springer, 2000)
-  John S. Wilson, *Profinite groups*, LMS Monographs **19** (OUP, 1998)