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A NEW CONSTRUCTION OF THE ASYMPTOTIC ALGEBRA ASSOCIATED TO THE q-SCHUR ALGEBRA

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ABSTRACT. We denote by A the ring of Laurent polynomials in the indeterminate v and by K its field of fractions. In this paper, we are interested in representation theory of the "generic" q-Schur algebra $\mathcal{S}_q(n,r)$ over A. We will associate to every non-degenerate symmetrising trace form τ on $K\mathcal{S}_q(n,r)$ a subalgebra $\mathcal{J}_\tau$ of $K\mathcal{S}_q(n,r)$ which is isomorphic to the "asymptotic" algebra $\mathcal{J}(n,r)_A$ defined by J. Du. As a consequence, we give a new criterion for James' conjecture.

1. INTRODUCTION

This article is concerned with the representation theory of the "generic" q-Schur algebra $\mathcal{S}_q(n,r)$ over $A=\mathbb{Z}[v,v^{-1}]$. The q-Schur algebra was introduced by Dipper and James in [3] and [4]. There is an interest in studying the representations of this algebra, because they relate informations about the modular representation theory of the finite general linear group $\mathrm{GL}_n(q)$ and of the quantum groups.

Using a new basis of $\mathcal{S}_q(n,r)$ constructed in [5] (which is analogous to the Kazhdan-Lusztig basis in Iwahori-Hecke algebras), J. Du introduced in [7] the asymptotic algebra $\mathcal{J}(n,r)_A$ over A and defined a homomorphism, $\Phi:\mathcal{S}_q(n,r)\to\mathcal{J}(n,r)_A$, the so-called Du-Lusztig homomorphism because its construction is similar to the Lusztig homomorphism for Iwahori-Hecke algebras.

There is a relevant open question in the representation theory of the q-Schur algebra, the so-called James' conjecture. A precise formulation of this conjecture is recalled in Section 6. In [9] Meinolf Geck obtained a new formulation of this conjecture. More precisely, for k any field of characteristic ℓ and for R any integral domain with quotient field k, if $q\in R$ is invertible, we can define the corresponding q-Schur algebra $\mathcal{S}_q(n,r)_R$ over R and its extension of scalars $\mathcal{S}_q(n,r)_k$. Similarly, we can define $\mathcal{J}(n,r)_k$.

In [9, 1.2] M. Geck has shown that James' conjecture holds if and only if, for $\ell>r$, the rank of the homomorphism $\Phi_k:\mathcal{S}_q(n,r)_k\to\mathcal{J}(n,r)_k$ only depends on the multiplicative order of q in $k^\times$, but not on ℓ.

Thus, in order to prove James' conjecture, it is relevant to understand the rank of the Du-Lusztig homomorphism. The motivation of this paper is to develop new methods allowing to study this rank. More precisely, we will give a new construction of the asymptotic algebra. Indeed, thanks to methods developed in [14] by the second author and adapted to our situation, we prove that $\mathcal{J}(n,r)_A$ is isomorphic to an algebra $\mathcal{J}_\tau$, which only depends on the choice of a non-degenerate symmetrising trace form τ on the semisimple algebra

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$K\mathcal{S}_q(n,r)$ (here $K=\mathbb{Q}(v)$) such that

$$\mathcal{S}_q(n,r) \subseteq \mathcal{J}_\tau \subseteq K\mathcal{S}_q(n,r).$$

Our main tool is to use the structure of the left cell modules of $\mathcal{S}_q(n,r)$ to construct an explicit Wedderburn basis of $K\mathcal{S}_q(n,r)$ (see Theorem 4.11). The main result of this paper is Theorem 5.5.

The article is organized as follows. In Section 2, we recall the definition of the "generic" q-Schur algebra and of its analogue of the Kazhdan-Lusztig basis for Iwahori-Hecke algebras. In Section 3 we prove that the q-Schur algebra satisfies properties which are very similar to Lusztigs conjectures **P1**,..., **P15** for Iwahori-Hecke algebras. In Section 4 we develop some tools to prove our main result in Section 5. Finally, in Section 6 we state a new criterion for James' conjecture.

2. THE IWAHORI-HECKE ALGEBRA OF TYPE A AND THE q-SCHUR ALGEBRA

Let v be an indeterminate. We set $A=\mathbb{Z}[v,v^{-1}]$ to be the ring of Laurent polynomials in v and $K:=\mathbb{Q}(v)$ its field of fractions. In order to introduce the q-Schur algebra over A, we have to recall some definitions and properties about Iwahori-Hecke algebras. We follow [13].

2.1. **Iwahori-Hecke algebras and the Kazhdan-Lusztig basis.** Let (W,S) be a Coxeter group (here S is the set of simple reflections). We define the corresponding Iwahori-Hecke algebra $\mathcal{H}$ as the free A-module with basis $\{T_w\}_{w\in W}$ satisfying

$$\begin{array}{ll} T_wT_{w'}=T_{ww'} & \text{if } l(ww')=l(w)+l(w'),\\ (T_s-v)(T_s+v^{-1})=0 & \text{for } s\in S,\end{array}$$

where l is the length function on W. In [12, §1] Kazhdan and Lusztig define an A-basis $\{C_w \mid w\in W\}$ of $\mathcal{H}$ which satisfies

$$\overline{C}_w=C_w \quad\text{and}\quad C_w=\sum_{y\leq w} p_{y,w}T_y \quad \text{for } w\in W,$$

where $\leq$ is the Bruhat-Chevalley order on W, and $^{-}:\mathcal{H}\to\mathcal{H}$ is the involutive automorphism of $\mathcal{H}$ defined by $\overline{v}=v^{-1}$ and $\overline{\sum_{w\in W} a_wT_w}=\sum_{w\in W}\overline{a}_wT_{w^{-1}}^{-1}$ and $p_{y,w}\in\langle v^k \mid k\leq 0\rangle_{\mathbb{Z}}$ and $p_{w,w}=1$.

Note that we use the more modern notation from [13], that is, our elements T_w here are the same as in [13] and were denoted by $v^{-l(w)}T_w$ in [12], and our elements C_w here were denoted by C'_w in [12] and by c_w in [13].

We denote by $g_{x,y,z}$ the structure constants of $\mathcal{H}$ with respect to the basis $\{C_w \mid w\in W\}$, that is, we have

$$C_xC_y=\sum_{z\in W} g_{x,y,z}C_z \quad \text{for } x,y\in W.$$

We define a relation $y \preccurlyeq_L w$ on W by: either $y=w$ or there is an $s\in S$ such that $g_{s,w,y}\neq 0$. Let $\leq_L$ be the transitive closure of the relation $\preccurlyeq_L$ and denote by $\sim_L$ the associated equivalence relation on W. The classes for this relation are the so-called left cells. Similarly, we define $\leq_R$ and $\sim_R$, and we call the corresponding equivalence classes right cells. For $y,w\in W$, we write $y\leq_{LR} w$ if there is a sequence $y=y_0, y_1\ldots,y_n=w$ of elements of W such that, for $i\in\{0,\ldots,n-1\}$, we have $y_i\leq_L y_{i+1}$ or $y_i\leq_R y_{i+1}$. The classes of the equivalence relation $\sim_{LR}$ on W corresponding to $\leq_{LR}$ are the so-called two-sided cells.

In [13, §3.6], Lusztig shows that for $z \in W$, there is a unique integer $\mathbf{a}(z)$ such that for every $x, y \in W$, we have $g_{x,y,z} \in v^{\mathbf{a}(z)}\mathbb{Z}[v^{-1}]$ and $g_{x,y,z} \notin v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}]$. Moreover, for $z \in W$, we define $\Delta(z) = -\deg p_{1,z}$. For $x, y, z \in W$, we write $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ for the coefficient of $v^{\mathbf{a}(z)}$ in $g_{x,y,z}$ and we set

$$\mathcal{D} = \{d \in W \mid \mathbf{a}(d) = \Delta(d)\},$$

the set of distinguished involutions. In the case that W is a finite Weyl group, an affine Weyl group, or a dihedral group, Lusztig proved that the following conjectures hold (see [13, §§15–17]):

- **P1** For any $z \in W$ we have $\mathbf{a}(z) \leq \Delta(z)$.
- **P2** Let $x,\, y \in W$; if $\gamma_{x,y,d} \neq 0$ for some $d \in \mathcal{D}$, then we have $x = y^{-1}$.
- **P3** If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.
- **P4** If $x \leq_{LR} y$, then $\mathbf{a}(x) \geq \mathbf{a}(y)$.
- **P5** If $d \in \mathcal{D}$ and $y \in W$ are such that $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = \pm 1$.
- **P6** For $d \in \mathcal{D}$, we have $d = d^{-1}$.
- **P7** For every $x,\, y,\, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y}$.
- **P8** Let $x,\, y,\, z \in W$ be such that $\gamma_{x,y,z} \neq 0$, then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$ and $z \sim_L x^{-1}$.
- **P9** If $x \leq_L y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_L y$.
- **P10** If $x \leq_R y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_R y$.
- **P11** If $x \leq_{LR} y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_{LR} y$.
- **P13** Every left cell contains a unique element $d \in \mathcal{D}$ and $\gamma_{y^{-1},y,d} \neq 0$ for every $y \sim_L d$.
- **P14** For every $x \in W$, we have $x \sim_{LR} x^{-1}$.
- **P15** Let v' be a second indeterminate and let $g'_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$ be obtained from $g_{x,y,z}$ by the substitution $v \mapsto v'$. If $x, x', y, w \in W$ satisfy $\mathbf{a}(w) = \mathbf{a}(y)$, then
$$\sum_{y'} g'_{w,x',y'} g_{x,y',y} = \sum_{y'} g_{x,w,y'} g'_{y',x',y}.$$

Note that in this paper we only consider the case of type A, in which W is the symmetric group on $|S| + 1$ points.

2.2. **The q-Schur algebra $\mathcal{S}_q(n, r)$.** In the following, we denote by W the symmetric group of degree r, and by S the set of transpositions $s_i = (i, i+1)$ for $1 \leq i \leq r-1$ and $\mathcal{H}$ is the associated Iwahori-Hecke algebra as in §2.1. Let $n, r \geq 1$, we denote by $\Lambda(n, r)$ the set of compositions of r into at most n parts. For $\lambda \in \Lambda(n, r)$, we denote by $W_\lambda \subseteq W$ the corresponding Young subgroup. For $\lambda, \mu \in \Lambda(n, r)$, we set $D_{\lambda,\mu}$ to be the set of distinguished double coset representatives of W with respect to W_λ and W_μ. We set

$$M(n, r) = \{(\lambda, w, \mu) \mid \lambda, \mu \in \Lambda(n, r),\, w \in D_{\lambda,\mu}\}.$$

For $\underline{a} = (\lambda, w, \mu) \in M(n, r)$, we write $ro(\underline{a}) = \lambda$ and $co(\underline{a}) = \mu$ and we set $\underline{a}^t = (\mu, w^{-1}, \lambda)$. For $\lambda,\, \mu \in \Lambda(n, r)$, we set $M_{\lambda,\mu} = \{\underline{a} \in M(n, r) \mid ro(\underline{a}) = \lambda,\, co(\underline{a}) = \mu\}$. We remark that if $w \in D_{\lambda,\mu}$, then the double coset $W_\lambda w W_\mu$ has a unique longest element. To prove this, we can proceed as follows: we denote by w_0 the longest element of W, then ${}^{w_0}W_\mu = W_{\widetilde{\mu}}$. Here $\widetilde{\mu} = (\mu_s, \mu_{s-1}, \ldots, \mu_1)$, where $\mu = (\mu_1, \ldots, \mu_s)$. Moreover, $r_{w_0} : W \to W,\, x \mapsto x w_0$ induces a bijection from the double coset $W_\lambda w w_0 W_{\widetilde{\mu}}$ to the double coset $W_\lambda w W_\mu$. Thanks to [13, 11.3], we deduce that r_{w_0} reverses the Bruhat-order. Since the double coset $W_\lambda w w_0 W_{\widetilde{\mu}}$ has a unique element of minimal length, the result follows. We write $D^+_{\lambda,\mu}$ for the set of double coset representatives of maximal length. We denote by $\ell_{\lambda,\mu}$ the bijection from $D_{\lambda,\mu}$ to $D^+_{\lambda,\mu}$ that associates to the representative of minimal length w of the double coset $W_\lambda w W_\mu$ the representative of maximal length. We remark that if $w \in D_{\lambda,\mu}$, then $w^{-1} \in D_{\mu,\lambda}$. Moreover, we have

$$\ell_{\lambda,\mu}(w)^{-1} = \ell_{\mu,\lambda}(w^{-1}).$$

In the following, we set $\sigma(\underline{a}) := \ell_{\lambda,\mu}(w)$ for $\underline{a} = (\lambda, w, \mu)$.

We now recall the definition of the q-Schur algebra $\mathcal{S}_q(n,r)$ introduced by Dipper and James in [3]. We set $q = v^2$, then the q-Schur algebra $\mathcal{S}_q(n,r)$ of degree (n,r) is the endomorphism algebra

$$\mathcal{S}_q(n,r) = \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H}\right),$$

where $x_\lambda = \sum_{w\in W_\lambda} v^{l(w)} T_w \in \mathcal{H}$. In [2, 3.4] Dipper and James prove that $\mathcal{S}_q(n,r)$ has a standard basis $\{\phi^w_{\lambda,\mu} \mid (\lambda,w,\mu) \in M(n,r)\}$ indexed by the set $M(n,r)$, which plays the same role as the basis $\{T_w \mid w \in W\}$ for the Iwahori-Hecke algebra $\mathcal{H}$. Moreover, in [5] Du proves that $\mathcal{S}_q(n,r)$ has another basis $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ whose construction is analogous to the Kazhdan-Lusztig basis of $\mathcal{H}$. We denote by $f_{\underline{a},\underline{b},\underline{c}} \in A$ the structure constants with respect to this basis, that is, we have

$$\theta_{\underline{a}}\theta_{\underline{b}} = \sum_{\underline{c}\in M(n,r)} f_{\underline{a},\underline{b},\underline{c}}\theta_{\underline{c}} \qquad \text{for all } \underline{a},\underline{b} \in M(n,r).$$

We recall the following lemma:

Lemma 2.3. *We have $f_{\underline{a},\underline{b},\underline{c}} \neq 0$ only if $co(\underline{a}) = ro(\underline{b})$ and $(ro(\underline{a}), co(\underline{b})) = (ro(\underline{c}), co(\underline{c}))$. In this case, we have*

$$f_{\underline{a},\underline{b},\underline{c}} = h_\mu^{-1} g_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c})}.$$

where $\mu = co(\underline{a}) = ro(\underline{b})$ and $h_\mu = \sum_{w\in W_\mu} v^{2l(w)-l(w_\mu)}$ (here w_μ denotes the longest element in W) and $g_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c})}$ is the structure constant of $\mathcal{H}$ defined in Section 2.1.

Proof. See [5, Prop. 3.4]. We want to explain why we have a further hypothesis here than in [5, Prop. 3.4]: For $\underline{a} = (\lambda, w, \mu) \in M(n,r)$ the element $\theta_{\underline{a}}$ is by definition a linear combination of basis elements $\phi^z_{\lambda,\mu}$ for $z \in \mathcal{D}_{\lambda,\mu}$. Thus, viewed as endomorphism of $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda\mathcal{H}$ it vanishes on all summands except on $x_\mu\mathcal{H}$ and maps into the summand $x_\lambda\mathcal{H}$. Thus, if either $co(\underline{a}) \neq ro(\underline{b})$ or $(ro(\underline{a}), co(\underline{b})) \neq (ro(\underline{c}), co(\underline{c}))$, the structure constant $f_{\underline{a},\underline{b},\underline{c}}$ vanishes also. If both equations hold, the proof in [5, Prop. 3.4] works using $g_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c})}$.

We are not claiming that [5, Prop. 3.4] is wrong as stated there. However, the notation $g_{\underline{a},\underline{b},\underline{c}}$ there needs proper interpretation (see [5, Section 3.3]), a problem we avoid here. □

Remark 2.4. To further explain the just mentioned change of notation, consider the following: Let $n = r = 3$, $\lambda := (2,1,0)$, $\mu := (1,1,1)$, and $\nu := (2,1,0)$. Then W is the symmetric group on 3 letters, generated by the two Coxeter generators $s_1 = (1,2)$ and $s_2 = (2,3)$. Thus $\mathcal{D}^+_{\lambda,\mu} := \{s_1, s_1s_2, s_1s_2s_1\}$, $\mathcal{D}^+_{\mu,\nu} = \{s_1, s_2s_1, s_1s_2s_1\}$ and $\mathcal{D}^+_{\lambda,\nu} = \{s_1, s_1s_2s_1\}$.

By the relations, we have $T_{s_1} \cdot T_{s_2s_1} = T_{s_1s_2s_1}$ and thus $g_{s_1,s_2s_1,s_1s_2s_1} = 1$. We now set $\underline{a} := (\lambda, \mathrm{id}, \mu)$, $\underline{b} := (\mu, s_2, \nu)$ and $\underline{c} := (\lambda, s_2, \nu)$. Thus, we get

$$f_{\underline{a},\underline{b},\underline{c}} = 1 \cdot g_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c})} = g_{s_1,s_2s_1,s_1s_2s_1} = 1,$$

since $h_\mu = 1$ here.

However, if we set $\underline{a'} := (\mu, s_1, \mu)$, then $f_{\underline{a'},\underline{b},\underline{c}} = 0$, because of $ro(\underline{a'}) \neq ro(\underline{c})$ and the arguments in the proof of Lemma 2.3. On the other hand, we have $ro(\underline{a'}) = co(\underline{b})$ and $g_{\sigma(\underline{a'}),\sigma(\underline{b}),\sigma(\underline{c})} = g_{s_1,s_2s_1,s_1s_2s_1} = 1$. This shows, that we indeed need all the hypothesis in Lemma 2.3. The statement in [5, Prop. 3.4] is true if one interprets $g_{\underline{a'},\underline{b},\underline{c}}$ to be zero.

Definition 2.5 (The $\mathbf{a}$-function and the distinguished elements)**.** Following [7, Section 2], we extend the $\mathbf{a}$-function to $M(n,r)$ by setting $\mathbf{a}(\underline{a}) = \mathbf{a}(\sigma(\underline{a}))$ for every $\underline{a} \in M(n,r)$ and we extend the set $\mathcal{D}$ to the set

$$\mathcal{D}(n,r) = \{\underline{d} \in M(n,r) \mid co(\underline{d}) = ro(\underline{d}),\ \sigma(\underline{d}) \in \mathcal{D}\}.$$

Moreover, for every $\underline{a},\ \underline{b},\ \underline{c} \in M(n,r)$, we define

$$\gamma_{\underline{a},\underline{b},\underline{c}^t} = \begin{cases} \gamma_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c}^t)} = \gamma_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c})^{-1}} & \text{if } f_{\underline{a},\underline{b},\underline{c}} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.6. Note that our definition for $\gamma_{\underline{a},\underline{b},\underline{c}}$ differs slightly from the one in [7, Section 2.2]. His $\gamma_{\underline{a},\underline{b},\underline{c}}$ is our $\gamma_{\underline{a},\underline{b},\underline{c}^t}$. With our definition we follow the setup in [13] more closely and get nicer cyclic symmetries in our formulas.

Remark 2.7. In comparison to [7, Section 2.1] we added the explicit hypothesis for the elements $\underline{d} \in \mathcal{D}(n,r)$ that $ro(\underline{d}) = co(\underline{d})$. However, this hypothesis is implicit in [7], since otherwise the statements in [7, 4.1,(a)–(d)] and some others would not be true.

Now, for $\underline{a},\ \underline{b} \in M(n,r)$, if there is $\underline{c} \in M(n,r)$ such that $f_{\underline{c},\underline{b},\underline{a}} \neq 0$ then we write $\underline{a} \leq_L \underline{b}$. We define $\leq_R$ by $\underline{a} \leq_R \underline{b}$ if and only if $\underline{a}^t \leq_L \underline{b}^t$. Moreover, we define $\leq_{LR}$ as in the Iwahori-Hecke algebra case. These relations induce corresponding equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$. We call the corresponding equivalence classes the left, right and two-sided cells of $M(n,r)$ respectively.

Let Γ be a left cell of $M(n,r)$. We set

$$\mathcal{S}_{\leq\Gamma} = \sum_{\underline{b}\leq_L\underline{a}} A\theta_{\underline{b}} \quad \text{and} \quad \mathcal{S}_{<\Gamma} = \sum_{\underline{b}\leq_L\underline{a},\ \underline{b}\not\sim_L\underline{a}} A\theta_{\underline{b}},$$

for some $\underline{a} \in \Gamma$, both are clearly left ideals of $\mathcal{S}_q(n,r)$ by the definition of $\leq_L$. Then the left cell module $\mathrm{LC}^{(\Gamma)}$ corresponding to Γ is defined as the quotient $\mathcal{S}_{\leq\Gamma}/\mathcal{S}_{<\Gamma}$.

We define the right cell module $\mathrm{RC}^{(\Gamma)}$ corresponding to a right cell Γ of $M(n,r)$ similarly. To see that we get right ideals we have to use Lemma 2.3 and $g_{x,y,z} = g_{y^{-1},x^{-1},z^{-1}}$ for $x, y, z \in W$ (see [13, 13.2.(e)]) together with $\sigma(\underline{a}^t) = \sigma(\underline{a})^{-1}$. This implies $f_{\underline{a},\underline{b},\underline{c}} = 0$ if and only if $f_{\underline{b}^t,\underline{a}^t,\underline{c}^t} = 0$.

3. Lusztig's conjectures for the q-Schur algebra

In this section, we prove that the q-Schur algebra satisfies properties very similar to **P1**, ..., **P15** for the Iwahori-Hecke algebra. First, we give some preliminary results.

Lemma 3.1. *If $\underline{a} \leq_L \underline{b}$ (resp. $\leq_R$, $\leq_{LR}$), then $\sigma(\underline{a}) \leq_L \sigma(\underline{b})$ (resp. $\leq_R$, $\leq_{LR}$).*

Proof. Since $\underline{a} \leq_L \underline{b}$, there is $\underline{c} \in M(n,r)$ such that $f_{\underline{c},\underline{b},\underline{a}} \neq 0$. But we have $f_{\underline{c},\underline{b},\underline{a}} = h^{-1}_{co(a)} g_{\sigma(\underline{c}),\sigma(\underline{b}),\sigma(\underline{a})}$ with $h^{-1}_{co(a)} \neq 0$. Thus $g_{\sigma(\underline{c}),\sigma(\underline{b}),\sigma(\underline{a})} \neq 0$ and $\sigma(\underline{a}) \leq_L \sigma(\underline{b})$. ☐

Lemma 3.2. *If $\underline{a} \leq_L \underline{b}$, then $co(\underline{a}) = co(\underline{b})$. If $\underline{a} \leq_R \underline{b}$, then $ro(\underline{a}) = ro(\underline{b})$.*

Proof. Since $\underline{a} \leq_L \underline{b}$ there is $\underline{c} \in M(n,r)$ such that $f_{\underline{c},\underline{b},\underline{a}} \neq 0$. From Lemma 2.3 follows that $(ro(\underline{a}), co(\underline{a})) = (ro(\underline{c}), co(\underline{b}))$ and the result is proved. ☐

Lemma 3.3. *Let $\lambda,\ \mu,\ \nu \in \Lambda(n,r)$, $x \in D^+_{\lambda,\mu}$ and $y \in D^+_{\mu,\nu}$. If $g_{x,y,z} \neq 0$ for some $z \in W$, then $z \in D^+_{\lambda,\nu}$.*

Proof. For $\lambda \in \Lambda(n,r)$ we set $S_\lambda := W_\lambda \cap S$, the set of Coxeter generators of the parabolic subgroup W_λ. Let $x \in \mathcal{D}^+_{\lambda,\mu}$ and $y \in \mathcal{D}^+_{\mu,\nu}$ and $g_{x,y,z} \neq 0$. On one hand, this means that $l(sx) < l(x)$ for all $s \in S_\lambda$ and $l(ys) < l(y)$ for all $s \in S_\nu$. On the other hand, we get $z \leq_L y$ and $z \leq_R x$ and thus $l(zs) < l(z)$ for all $s \in S$ with $l(ys) < l(y)$ and $l(sz) < l(s)$ for all $s \in S$ with $l(sx) < l(x)$ by [13, Lemma 8.6]. Thus we have in particular that $l(zs) < l(z)$ for all $s \in S_\nu$ and $l(sz) < l(z)$ for all $s \in S_\lambda$. Hence z is the longest element in its W_λ-W_ν-double coset in W. □

Lemma 3.4. *We have $\underline{a} \leq_R \underline{b}$ if and only if there is a $\underline{c} \in M(n,r)$ with $f_{\underline{b},\underline{c},\underline{a}} \neq 0$.*

Proof. By definition, $\underline{a} \leq_R \underline{b}$ is equivalent to $\underline{a}^t \leq_L \underline{b}^t$. This in turn means that there is a $\underline{c} \in M(n,r)$ such that $f_{\underline{c}^t,\underline{b}^t,\underline{a}^t} \neq 0$. As mentioned at the end of Section 2.2 we have $f_{\underline{b},\underline{c},\underline{a}} = 0$ if and only if $f_{\underline{c}^t,\underline{b}^t,\underline{a}^t} = 0$ which directly implies the statement in the lemma. □

Proposition 3.5. *The following properties hold for the q-Schur algebra:*

- ***Q1*** *For any $\underline{a} \in M(n,r)$ we have $\mathbf{a}(\underline{a}) \leq \Delta(\sigma(\underline{a}))$.*
- ***Q2*** *If $\gamma_{\underline{a},\underline{b},\underline{d}} \neq 0$ for some $\underline{d} \in \mathcal{D}(n,r)$, then we have $\underline{b} = \underline{a}^t$.*
- ***Q3*** *For every $\underline{a} \in M(n,r)$, there is a unique $\underline{d} \in \mathcal{D}(n,r)$ with $\gamma_{\underline{a}^t,\underline{a},\underline{d}} \neq 0$.*
- ***Q4*** *If $\underline{a} \leq_{LR} \underline{b}$, then $\mathbf{a}(\underline{a}) \geq \mathbf{a}(\underline{b})$.*
- ***Q5*** *If $\underline{d} \in \mathcal{D}(n,r)$ and $\underline{a} \in M(n,r)$ are such that $\gamma_{\underline{a}^t,\underline{a},\underline{d}} \neq 0$, then $\gamma_{\underline{a}^t,\underline{a},\underline{d}} = 1$.*
- ***Q6*** *For $\underline{d} \in \mathcal{D}(n,r)$, we have $\underline{d} = \underline{d}^t$.*
- ***Q7*** *For every $\underline{a},\ \underline{b},\ \underline{c} \in M(n,r)$, we have $\gamma_{\underline{a},\underline{b},\underline{c}} = \gamma_{\underline{b},\underline{c},\underline{a}} = \gamma_{\underline{c},\underline{a},\underline{b}}$.*
- ***Q8*** *Let $\underline{a},\ \underline{b},\ \underline{c} \in M(n,r)$ be such that $\gamma_{\underline{a},\underline{b},\underline{c}} \neq 0$, then $\underline{a} \sim_L \underline{b}^t$, $\underline{b} \sim_L \underline{c}^t$ and $\underline{c} \sim_L \underline{a}^t$.*
- ***Q9*** *If $\underline{a} \leq_L \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$, then $\underline{a} \sim_L \underline{b}$.*
- ***Q10*** *If $\underline{a} \leq_R \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$, then $\underline{a} \sim_R \underline{b}$.*
- ***Q11*** *If $\underline{a} \leq_{LR} \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$, then $\underline{a} \sim_{LR} \underline{b}$.*
- ***Q13*** *Every left cell contains a unique element $\underline{d} \in \mathcal{D}(n,r)$ and $\gamma_{\underline{a}^t,\underline{a},\underline{d}} \neq 0$ for every $\underline{a} \sim_L \underline{d}$.*
- ***Q14*** *For every $\underline{a} \in M(n,r)$, we have $\underline{a} \sim_{LR} \underline{a}^t$.*
- ***Q15*** *Let v' be a second indeterminate and let $f'_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$ be obtained from $f_{x,y,z}$ by the substitution $v \mapsto v'$. If $\underline{a}, \underline{a}', \underline{b}, \underline{c} \in W$ satisfy $\mathbf{a}(\underline{c}) = \mathbf{a}(\underline{b})$, then*

$$\sum_{\underline{b}'} f'_{\underline{c},\underline{a}',\underline{b}'} f_{\underline{a},\underline{b}',\underline{b}} = \sum_{\underline{b}'} f_{\underline{a},\underline{c},\underline{b}'} f'_{\underline{b}',\underline{a}',\underline{b}}.$$

Proof. We note that **Q1** is a direct consequence of Property **P1**.

We now will prove Property **Q2**. We suppose that $\gamma_{\underline{a},\underline{b},\underline{d}} \neq 0$ for some $\underline{a},\ \underline{b} \in M(n,r)$ and $\underline{d} \in \mathcal{D}(n,r)$. Since $\gamma_{\underline{a},\underline{b},\underline{d}} \neq 0$, it follows that $f_{\underline{a},\underline{b},\underline{d}} \neq 0$. Thus we have $co(\underline{a}) = ro(\underline{b})$, $ro(\underline{a}) = ro(\underline{d})$ and $co(\underline{b}) = co(\underline{d})$ by Lemma 2.3. But $co(\underline{d}) = ro(\underline{d})$ implies $ro(\underline{a}) = co(\underline{b})$. We now write $\underline{a} = (\lambda, w_a, \mu)$ and $\underline{b} = (\mu, w_b, \lambda)$. We have $\gamma_{\underline{a},\underline{b},\underline{d}} = \gamma_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{d})}$. From $\sigma(\underline{d}) \in \mathcal{D}$ we deduce using **P2** that $\sigma(\underline{a}) = \sigma(\underline{b})^{-1}$. It follows that $\ell_{\lambda,\mu}(w_a) = \ell_{\mu,\lambda}(w_b)^{-1} = \ell_{\lambda,\mu}(w_b^{-1})$, we get $w_a = w_b^{-1}$ and thus **Q2** holds.

Let $\underline{a} = (\lambda, w, \mu) \in M(n,r)$. Thanks to Property **P3**, there is a unique $d \in \mathcal{D}$ such that $\gamma_{\sigma(\underline{a})^{-1},\sigma(\underline{a}),d} \neq 0$. Since $\sigma(\underline{a})^{-1} = \sigma(\underline{a}^t)$, we deduce that $g_{\sigma(\underline{a}^t),\sigma(\underline{a}),d} \neq 0$. But $\sigma(\underline{a}^t) \in D^+_{\mu,\lambda}$ and $\sigma(\underline{a}) \in D^+_{\lambda,\mu}$, then Lemma 3.3 gives $d \in D^+_{\mu,\mu}$. We denote by $\widetilde{d}$ the representative of minimal length of the coset $W_\mu d W_\mu$ and we set $\underline{d} := (\mu, \widetilde{d}, \mu)$. Then $\underline{d} \in \mathcal{D}(n,r)$ and $\sigma(\underline{d}) = d$. It follows that $\gamma_{\underline{a}^t,\underline{a},\underline{d}} \neq 0$ and thus **Q3** holds.

The property **Q4** follows from **P4** and Lemma 3.1. The property **Q5** directly follows from **P5**, since in our case W is of type A and thus all coefficients of all Kazhdan-Lusztig polynomials are non-negative by [13, 15.1].

Let $\underline{d} = (\lambda, w, \lambda) \in \mathcal{D}(n,r)$; we have $\sigma(\underline{d}) \in \mathcal{D}$, thus **P6** gives $\sigma(\underline{d})^{-1} = \sigma(\underline{d})$. Therefore, we have $\ell_{\lambda,\lambda}(w) = \sigma(\underline{d})^{-1} = \sigma(\underline{d}^t) = \ell_{\lambda,\lambda}(w^{-1})$, and it follows that $w = w^{-1}$; thus **Q6** holds. The property **Q7** follows directly from **P7**.

Suppose that $\gamma_{\underline{a},\underline{b},\underline{c}} \neq 0$ for some $\underline{a}$, $\underline{b}$, $\underline{c} \in M(n,r)$, then $f_{\underline{a},\underline{b},\underline{c}^t} \neq 0$ and it follows that $co(\underline{a}) = ro(\underline{b})$ and $(ro(\underline{a}), co(\underline{b})) = (ro(\underline{c}^t), co(\underline{c}^t))$. Then we have

$$\begin{aligned} f_{\underline{b}^t,\underline{a}^t,\underline{c}} &= h_{co(\underline{a})} g_{\sigma(\underline{b}^t),\sigma(\underline{a}^t),\sigma(\underline{c})} \\ &= h_{co(\underline{a})} g_{\sigma(\underline{b})^{-1},\sigma(\underline{a})^{-1},\sigma(\underline{c})} \\ &= h_{co(\underline{a})} g_{\sigma(\underline{a}),\sigma(\underline{b}),\sigma(\underline{c})^{-1}} \\ &= f_{\underline{a},\underline{b},\underline{c}^t}. \end{aligned}$$

It follows that $\underline{c}^t \leq_L \underline{b}$ and $\underline{c} \leq_L \underline{a}^t$. Using **Q7** and the same arguments applied to $\gamma_{\underline{b},\underline{c},\underline{a}} = \gamma_{\underline{c},\underline{a},\underline{b}} \neq 0$, we deduce that $\underline{a} \sim_L \underline{b}^t$, $\underline{b} \sim_L \underline{c}^t$ and $\underline{c} \sim_L \underline{a}^t$. Thus **Q8** holds.

Next we prove **Q13**. Let $\underline{a} \in M(n,r)$. By **Q3** there is a unique $\underline{d} \in \mathcal{D}(n,r)$ with $\gamma_{\underline{a}^t,\underline{a},\underline{d}} \neq 0$ and for this $\underline{d}$ holds $\underline{a} \sim_L \underline{d}$ by **Q8**. But for $\underline{d}, \underline{d}' \in \mathcal{D}(n,r)$ with $\underline{d} \sim_L \underline{d}'$ we conclude $ro(\underline{d}) = co(\underline{d}) = co(\underline{d}') = ro(\underline{d}')$ using Lemma 3.2 and $\sigma(\underline{d}) = \sigma(\underline{d}')$ using **P13** since $\sigma(\underline{d}) \sim_L \sigma(\underline{d}')$ because of Lemma 3.1. Thus we have proved **Q13**.

Now we prove **Q9**. Let $\underline{a}, \underline{b} \in M(n,r)$ with $\underline{a} \leq_L \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$. We denote the unique element of $\mathcal{D}(n,r)$ in the left cell of $\underline{a}$ by $\underline{d}_a$ (resp. $\underline{d}_b$ for $\underline{b}$). Using **Q4** we deduce that $\mathbf{a}(\underline{d}_a) = \mathbf{a}(\underline{a})$ and $\mathbf{a}(\underline{d}_b) = \mathbf{a}(\underline{b})$. Moreover, we have $\underline{d}_a \leq_L \underline{d}_b$. Thus using Lemma 3.1 shows that $\sigma(\underline{d}_a) \leq_L \sigma(\underline{d}_b)$. Hence, using Property **P9**, we have $\sigma(\underline{d}_a) \sim_L \sigma(\underline{d}_b)$. However, $\sigma(\underline{d}_a)$ and $\sigma(\underline{d}_b)$ lie in $\mathcal{D}$. Therefore, using **P13** in the Iwahori-Hecke algebra, we deduce that $\sigma(\underline{d}_a) = \sigma(\underline{d}_b)$. We now prove that $f_{\underline{d}_a,\underline{d}_a,\underline{d}_b} \neq 0$. Since $ro(\underline{d}_a) = co(\underline{d}_a) = co(\underline{d}_b) = ro(\underline{d}_b)$ (thanks to Lemma 3.2), we deduce that

$$f_{\underline{d}_a,\underline{d}_a,\underline{d}_b} = h^{-1}_{co(\underline{d}_a)} g_{\sigma(\underline{d}_a),\sigma(\underline{d}_a),\sigma(\underline{d}_b)}.$$

Using **P13**, we deduce that $\gamma_{\sigma(\underline{d}_a)^{-1},\sigma(\underline{d}_a),\sigma(\underline{d}_b)} \neq 0$; hence $g_{\sigma(\underline{d}_a),\sigma(\underline{d}_a),\sigma(\underline{d}_b)} \neq 0$. Since $h^{-1}_{co(\underline{d}_a)} \neq 0$, it follows that $f_{\underline{d}_a,\underline{d}_a,\underline{d}_b} \neq 0$. Hence $\underline{d}_b \leq_L \underline{d}_a$ and **Q9** follows.

Property **Q10** follows from **Q9** by transposition since $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{a}^t)$ for all $\underline{a} \in M(n,r)$ (use [13, 13.9 (a)]). Property **Q11** follows from **Q9** and **Q10** and induction.

Let $\underline{a} \in M(n,r)$ and $\underline{d} \in \mathcal{D}(n,r)$ be the unique element such that $\underline{a} \sim_L \underline{d}$ given by **Q13**. Then $\underline{a}^t \sim_R \underline{d}^t = \underline{d}$ and **Q14** holds.

Finally, we prove **Q15**. We first remark that $f'_{\underline{c},\underline{a}',\underline{b}'} \neq 0$ if and only if $f_{\underline{a},\underline{c},\underline{b}'} \neq 0$, and $f_{\underline{a},\underline{b}',\underline{b}} \neq 0$ if and only if $f'_{\underline{b}',\underline{a}',\underline{b}} \neq 0$. Moreover if $f'_{\underline{c},\underline{a}',\underline{b}'} \neq 0$, then $f'_{\underline{c},\underline{a}',\underline{b}'} = h'_{ro(\underline{a}')} g_{\sigma(\underline{c}),\sigma(\underline{a}'),\sigma(\underline{b}')}$ and $f_{\underline{a},\underline{c},\underline{b}'} = h_{co(\underline{a})} g_{\sigma(\underline{a}),\sigma(\underline{c}),\sigma(\underline{b}')}$. If $f_{\underline{a},\underline{c},\underline{b}'} \neq 0$, then $f_{\underline{a},\underline{c},\underline{b}'} = h_{co(\underline{a})} g_{\sigma(\underline{a}),\sigma(\underline{c}),\sigma(\underline{b}')}$ and $f'_{\underline{b}',\underline{a}',\underline{b}} = h'_{ro(\underline{a}')} g_{\sigma(\underline{b}'),\sigma(\underline{a}'),\sigma(\underline{b})}$. Here h'_μ is obtained from h_μ by the substitution $v \mapsto v'$. We note that $h_{ro(\underline{a}')}$ and $h_{co(\underline{a})}$ do not depend on $\underline{b}'$. It follows from **P15** that

$$\begin{aligned} \sum_{\underline{b}'} f'_{\underline{c},\underline{a}',\underline{b}'} f_{\underline{a},\underline{b}',\underline{b}} &= h_{ro(\underline{a}')} h_{co(\underline{a})} \sum_{\underline{b}'} g'_{\sigma(\underline{c}),\sigma(\underline{a}'),\sigma(\underline{b}')} g_{\sigma(\underline{a}),\sigma(\underline{b}'),\sigma(\underline{b})} \\ &= h_{ro(\underline{a}')} h_{co(\underline{a})} \sum_{\underline{b}'} g_{\sigma(\underline{a}),\sigma(\underline{c}),\sigma(\underline{b}')} f'_{\sigma(\underline{b}'),\sigma(\underline{a}'),\sigma(\underline{b})} \\ &= \sum_{\underline{b}'} f_{\underline{a},\underline{c},\underline{b}'} f'_{\underline{b}',\underline{a}',\underline{b}}. \end{aligned}$$

□

Proposition 3.6. *If $\underline{a} \sim_L \underline{b}$ and $\underline{a} \sim_R \underline{b}$, then $\underline{a} = \underline{b}$.*

Proof. Let $\underline{a} = (\lambda_a, w_a, \mu_a)$ and $\underline{b} = (\lambda_b, w_b, \mu_b)$ be such that $\underline{a} \sim_L \underline{b}$ and $\underline{a} \sim_R \underline{b}$. We have $\underline{a} \leq_L \underline{b}$ and $\underline{a}^t \leq_L \underline{b}^t$, then using Lemma 3.2 we deduce that $\mu_a = \mu_b$ and $\lambda_a = \lambda_b$. Using Lemma 3.1, we deduce that $\sigma(\underline{a}) \sim_L \sigma(\underline{b})$ and $\sigma(\underline{a}) \sim_R \sigma(\underline{b})$. Since $\mathcal{H}$ is of type A, it follows that $\sigma(\underline{a}) = \sigma(\underline{b})$, that is $\ell_{\lambda_a,\mu_a}(w_a) = \ell_{\lambda_a,\mu_a}(w_b) = \ell_{\lambda_b,\mu_b}(w_b)$. Hence we get $w_a = w_b$. □

4. Irreducible cell modules and dual basis

In this section we view the extension of scalars $K\mathcal{S}_q(n,r)$ of the q-Schur algebra $\mathcal{S}_q(n,r)$ as a symmetric algebra. This is possible, since it is semisimple (see [1, (9.8)]). We can take as symmetrising trace form any K-linear form $\tau : K\mathcal{S}_q(n,r) \to K$ that is a K-linear combination

$$\tau = \sum_{\chi \in \mathrm{Irr}(K\mathcal{S}_q(n,r))} \frac{\chi}{c_\chi}$$

of the irreducible characters where the c_χ are non-zero constants, the so-called Schur elements (see [10, 7.1.1 and 7.2.6]). Clearly, τ is non-degenerate.

Having fixed τ, we denote for any K-basis $(B_{\underline{a}})_{\underline{a} \in M(n,r)}$ of $K\mathcal{S}_q(n,r)$ its dual basis with respect to τ by $(B^\vee_{\underline{b}})_{\underline{b} \in M(n,r)}$. That is, we have $\tau(B_{\underline{a}} \cdot B^\vee_{\underline{b}}) = \tau(B^\vee_{\underline{b}} \cdot B_{\underline{a}}) = \delta_{\underline{a},\underline{b}}$ for all $\underline{a}, \underline{b} \in M(n,r)$. Note that this immediately implies that we can write every element $x \in K\mathcal{S}_q(n,r)$ in the following form:

$$x = \sum_{\underline{a} \in M(n,r)} \tau(x \cdot B^\vee_{\underline{a}}) B_{\underline{a}} = \sum_{\underline{a} \in M(n,r)} \tau(x \cdot B_{\underline{a}}) B^\vee_{\underline{a}} \tag{4.1}$$

(just write x as a linear combination of the $B_{\underline{a}}$, multiply by some $B_{\underline{b}}$ and apply τ).

Remark 4.1. We have $f_{\underline{a},\underline{b},\underline{c}} = \tau(\theta_{\underline{a}} \cdot \theta_{\underline{b}} \cdot \theta^\vee_{\underline{c}})$ for all $\underline{a}, \underline{b}, \underline{c} \in M(n,r)$. Moreover, we note that Formula (4.1) immediately gives us nice formulas for the matrix representations coming from the left cell modules. For a left cell Γ and an element $h \in \mathcal{S}_q(n,r)$ the representing matrix of h on the left cell module $\mathrm{LC}^{(\Gamma)}$ with respect to the basis $\{\theta_{\underline{a}} + \mathcal{S}_{<\Gamma} \mid \underline{a} \in \Gamma\}$ is $\left(\tau(\theta^\vee_{\underline{b}} \cdot h \cdot \theta_{\underline{a}})\right)_{\underline{b},\underline{a} \in \Gamma}$ since $h \cdot \theta_{\underline{a}} = \sum_{\underline{b} \in M(n,r)} \tau(\theta^\vee_{\underline{b}} \cdot h \cdot \theta_{\underline{a}}) \cdot \theta_{\underline{b}}$ and it is enough to sum over those $\underline{b}$ with $\underline{b} \leq_L \underline{a}$.

Lemma 4.2 (Characterisation of $\leq_L$ and $\leq_R$)**.** *We have $\underline{a} \leq_L \underline{b}$ if and only if $\theta_{\underline{b}} \theta^\vee_{\underline{a}} \neq 0$ and $\underline{a} \leq_R \underline{b}$ if and only if $\theta^\vee_{\underline{a}} \theta_{\underline{b}} \neq 0$.*

Proof. We only show the version with $\leq_L$, the other is completely analogous thanks to Lemma 3.4. If $\underline{a} \leq_L \underline{b}$ there exists a $\underline{c} \in M(n,r)$ with $f_{\underline{c},\underline{b},\underline{a}} = \tau(\theta_{\underline{c}} \theta_{\underline{b}} \theta^\vee_{\underline{a}}) \neq 0$ which implies $\theta_{\underline{b}} \theta^\vee_{\underline{a}} \neq 0$. If we assume the latter, then by the non-degeneracy of τ there is some $\underline{c} \in M(n,r)$ with $\tau(\theta_{\underline{c}} \theta_{\underline{b}} \theta^\vee_{\underline{a}}) \neq 0$ and $\underline{a} \leq_L \underline{b}$ follows. □

The other major ingredient is the fact that cell modules are simple, more precisely:

Theorem 4.3 (Simple cell modules, see [6] or [7, 4.3])**.** *Let Γ be a left cell and recall $K = \mathbb{Q}(v)$. The extension of scalars $K\,\mathrm{LC}^{(\Gamma)}$ of the left cell module $\mathrm{LC}^{(\Gamma)}$ for a left cell Γ is a simple $K\mathcal{S}_q(n,r)$-module.*

Proof. See [6] or [7, 4.3]. □

Remark 4.4. This in particular implies that all simple $K\mathcal{S}_q(n,r)$-modules can be realised over the ring A, since their corresponding representating matrices involve only structure constants of $\mathcal{S}_q(n,r)$.

We now directly obtain useful algebra elements by using the simple cell modules:

Theorem 4.5 (Basis of an isotypic component)**.** *Let Γ be a left cell and χ the corresponding irreducible character of the left cell module $\mathrm{LC}^{(\Gamma)}$, then the elements*

$$\left(c_\chi^{-1}\theta_{\underline{a}}\,\theta_{\underline{b}}^\vee\right)_{\underline{a},\underline{b}\in\Gamma}$$

are K-linearly independent and span the isotypic component of $K\mathcal{S}_q(n,r)$ belonging to the character χ. Furthermore, we have the relations

$$\left(c_\chi^{-1}\theta_{\underline{a}}\,\theta_{\underline{b}}^\vee\right)\cdot\left(c_\chi^{-1}\theta_{\underline{a}'}\theta_{\underline{b}'}^\vee\right)=\delta_{\underline{b},\underline{a}'}\cdot c_\chi^{-1}\theta_{\underline{a}}\,\theta_{\underline{b}'}^\vee$$

for all $\underline{a},\underline{b},\underline{a}',\underline{b}'\in\Gamma$. That is, these elements form a matrix unit for the isotypic component of $K\mathcal{S}_q(n,r)$ corresponding to the simple module $K\,\mathrm{LC}^{(\Gamma)}$.

Proof. By [10, 7.2.7] we get a matrix unit for the isotypic component of $K\mathcal{S}_q(n,r)$ corresponding to the simple module $K\,\mathrm{LC}^{(\Gamma)}$ by the elements

$$\frac{1}{c_\chi}\sum_{\underline{c}\in M(n,r)}\tau(\theta_{\underline{b}}^\vee\cdot\theta_{\underline{c}}\cdot\theta_{\underline{a}})\cdot\theta_{\underline{c}}^\vee=\frac{1}{c_\chi}\sum_{\underline{c}\in M(n,r)}\tau(\theta_{\underline{c}}\cdot\theta_{\underline{a}}\theta_{\underline{b}}^\vee)\cdot\theta_{\underline{c}}^\vee$$

for $\underline{a},\underline{b}\in\Gamma$. But this is equal to $c_\chi^{-1}\theta_{\underline{a}}\theta_{\underline{b}}^\vee$ by Formula (4.1). □

Corollary 4.6. *Let Γ be a left cell and χ the corresponding irreducible character of the left cell module $\mathrm{LC}^{(\Gamma)}$. Then the element*

$$e_\Gamma:=\frac{1}{c_\chi}\sum_{\underline{a}\in\Gamma}\theta_{\underline{a}}\theta_{\underline{a}}^\vee$$

is the central primitive idempotent of $K\mathcal{S}_q(n,r)$ corresponding to the irreducible character χ.

Proof. By Theorem 4.5 e_Γ lies in the isotypic component corresponding to the character χ and is mapped to the identity matrix in the corresponding matrix representation. □

Lemma 4.7 (Isomorphism of left cell modules and two-sided cells)**.** *Let Γ and Γ' be left cells. If $K\,\mathrm{LC}^{(\Gamma)}$ and $K\,\mathrm{LC}^{(\Gamma')}$ are isomorphic $K\mathcal{S}_q(n,r)$-modules then Γ and Γ' lie in the same two-sided cell.*

Proof. Let χ be the irreducible character of the left cell module $\mathrm{LC}^{(\Gamma)}$ and χ' that of $\mathrm{LC}^{(\Gamma')}$. The modules $K\,\mathrm{LC}^{(\Gamma)}$ and $K\,\mathrm{LC}^{(\Gamma')}$ are isomorphic if and only if $e_\Gamma\cdot e_{\Gamma'}=e_{\Gamma'}\cdot e_\Gamma\neq 0$ (and in this case $e_\Gamma=e_{\Gamma'}$). Now assume this case. Then

$$0\neq\frac{1}{c_\chi^2}\sum_{\underline{a}\in\Gamma}\sum_{\underline{b}\in\Gamma'}\theta_{\underline{a}}\theta_{\underline{a}}^\vee\theta_{\underline{b}}\theta_{\underline{b}}^\vee=\frac{1}{c_\chi^2}\sum_{\underline{a}\in\Gamma}\sum_{\underline{b}\in\Gamma'}\theta_{\underline{b}}\theta_{\underline{b}}^\vee\theta_{\underline{a}}\theta_{\underline{a}}^\vee$$

and thus there is at least one pair $(\underline{a},\underline{b})\in\Gamma\times\Gamma'$ such that $\theta_{\underline{a}}^\vee\theta_{\underline{b}}\neq 0$. By Lemma 4.2 this implies $\underline{a}\leq_R\underline{b}$. Since e_Γ and $e_{\Gamma'}$ commute, the same argument shows $\underline{b}'\leq_R\underline{a}'$ for some $\underline{a}'\in\Gamma$ and $\underline{b}'\in\Gamma'$. Thus, Γ and Γ' lie in the same two-sided cell in that case. □

For what follows we need the following statement about Iwahori-Hecke-Algebras of type A:

Theorem 4.8 (Equal cell modules in the Iwahori-Hecke algebra)**.** *Let $\mathcal{H}$ be a generic Iwahori-Hecke-Algebra of type A as in Section 2. If $x \sim_L y$ and $z \sim_L w$ and $x \sim_R z$ and $y \sim_R w$, then $C_x D_{y^{-1}} = C_z D_{w^{-1}}$. In particular, we have*

$$g_{u,x,y} = \tau(C_u C_x D_{y^{-1}}) = \tau(C_u C_z D_{w^{-1}}) = g_{u,z,w}$$

for all $u \in W$.

Proof. This statement is already implicitly stated in [12]. Namely, it is shown there in the proof of Theorem 1.4 that the two left cell modules defined by the left cell containing x, y and the one containing z, w are isomorphic since all four lie in the same two-sided. The exact statement there is that two W-graphs are isomorphic, which means in particular that not only the two left cell modules are isomorphic, but that even the matrix representations with respect to the bases $\{C_v \mid v \sim_L x\}$ and $\{C_w \mid w \sim_L z\}$ are equal. But this exactly means, that

$$\tau(D_{y^{-1}} C_u C_x) = \tau(D_{w^{-1}} C_u C_z)$$

for all $u \in W$ which we claim. □

Now we begin to use statements **Q1** to **Q14**:

Theorem 4.9 (Equality of different left cell modules)**.** *Let Γ, Γ' be left cells such that $K\,\mathrm{LC}^{(\Gamma)}$ and $K\,\mathrm{LC}^{(\Gamma')}$ are isomorphic $K\mathcal{S}_q(n,r)$-modules. Let $\underline{d}$ be the unique element in $\Gamma' \cap \mathcal{D}(n,r)$ (use **Q13**) and $\underline{c} \sim_L \underline{d}$ that is $\underline{c} \in \Gamma'$. Then there are unique $\underline{a}, \underline{b} \in \Gamma$ with $\underline{a} \sim_R \underline{c}$ and $\underline{b} \sim_R \underline{d}$ and we have $\theta_{\underline{a}} \theta_{\underline{b}}^\vee = \theta_{\underline{c}} \theta_{\underline{d}}^\vee$.*

Proof. Let χ be the irreducible character of the left cell module $\mathrm{LC}^{\Gamma'}$. We denote by c_χ the corresponding Schur element. Since $\underline{c} \sim_L \underline{d}$, it follows from Theorem 4.5 that

$$\theta_{\underline{d}}\, \theta_{\underline{c}}^\vee \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee = c_\chi \theta_{\underline{d}}\, \theta_{\underline{d}}^\vee.$$

Therefore we have $\tau(\theta_{\underline{c}}^\vee \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee \theta_{\underline{d}}) \neq 0$ and hence $\theta_{\underline{c}}\, \theta_{\underline{d}}^\vee$ acts non-trivially on the module $\mathrm{LC}^{(\Gamma')}$ (see Remark 4.1) and thus also on the isomorphic module $\mathrm{LC}^{(\Gamma)}$.

This means that there is at least one pair $(\underline{a}, \underline{b}) \in \Gamma \times \Gamma$ such that

$$\tau(\theta_{\underline{b}}\, \theta_{\underline{a}}^\vee \cdot \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee) = \tau(\theta_{\underline{a}}^\vee \cdot \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee \cdot \theta_{\underline{b}}) = \tau(\theta_{\underline{c}}\, \theta_{\underline{d}}^\vee \cdot \theta_{\underline{b}}\, \theta_{\underline{a}}^\vee) \neq 0.$$

But then in particular $\theta_{\underline{a}}^\vee \theta_{\underline{c}} \neq 0$ and thus $a \leq_R c$ by Lemma 4.2. Since Γ and Γ' lie in the same two-sided cell by Lemma 4.7, we conclude $\underline{a} \sim_{LR} \underline{c}$ and thus by **Q4** and **Q10** $\underline{a} \sim_R \underline{c}$. Analogously, we show $\underline{b} \sim_R \underline{d}$. By Proposition 3.6 we conclude that there is only one such pair $(\underline{a}, \underline{b})$ since both are uniquely defined by their membership in a left and a right cell.

We now show that $f_{\underline{e},\underline{a},\underline{b}} = f_{\underline{e},\underline{c},\underline{d}}$ for all $\underline{e} \in M(n,r)$ and thus $\theta_{\underline{a}}\, \theta_{\underline{b}}^\vee = \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee$. We have $co(\underline{a}) = co(\underline{b})$ and $co(\underline{c}) = co(\underline{d}) = ro(\underline{d}) = ro(\underline{b})$ and $ro(\underline{a}) = ro(\underline{c})$ by Lemma 3.2 and the fact that $\underline{d} \in \mathcal{D}(n,r)$. Thus, if $ro(\underline{e}) \neq ro(\underline{b})$ or $co(\underline{e}) \neq ro(\underline{a})$ then both sides are zero by Lemma 2.3. Otherwise, we have

$$f_{\underline{e},\underline{a},\underline{b}} = h_{co(\underline{e})}^{-1} \cdot g_{\sigma(\underline{e}),\sigma(\underline{a}),\sigma(\underline{b})} \qquad \text{and} \qquad f_{\underline{e},\underline{c},\underline{d}} = h_{co(\underline{e})}^{-1} \cdot g_{\sigma(\underline{e}),\sigma(\underline{c}),\sigma(\underline{d})}$$

and thus the equality $f_{\underline{e},\underline{a},\underline{b}} = f_{\underline{e},\underline{c},\underline{d}}$ follows from

$$\sigma(\underline{a}) \sim_L \sigma(\underline{b}) \sim_R \sigma(\underline{d}) \sim_L \sigma(\underline{c}) \sim_R \sigma(\underline{a})$$

using Lemma 3.1 and Theorem 4.8. The non-degeneracy of τ now immediately implies $\theta_{\underline{a}}\, \theta_{\underline{b}}^\vee = \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee$. □

With this we get the following result, for which we first need one more piece of notation:

Definition 4.10 (Schur elements of characters of left cell modules)**.** Let $\underline{d} \in \mathcal{D}(n,r)$ and Γ the unique left cell with $\underline{d} \in \Gamma$ (remember **Q13**). We denote the left cell module $\mathrm{LC}^{(\Gamma)}$ by $\mathrm{LC}^{(\underline{d})}$ and the Schur element corresponding to the irreducible character of $\mathrm{LC}^{(\underline{d})}$ by $c_{\underline{d}}$.

Theorem 4.11 (Wedderburn basis)**.** *Let τ be an arbitrary non-degenerate symmetrising trace form on $K\mathcal{S}_q(n,r)$. The set*

$$\mathcal{B} := \{c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee} \mid \underline{c} \in M(n,r), \underline{d} \in \mathcal{D}(n,r), \underline{c} \sim_L \underline{d}\}$$

is a Wedderburn basis of $K\mathcal{S}_q(n,r)$. Two elements $c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}$ and $c_{\underline{d'}}^{-1}\theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee}$ lie in the same isotypic component if and only if $\mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d'})}$.

For $c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}, c_{\underline{d'}}^{-1}\theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee} \in \mathcal{B}$ we have the following equation:

$$c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee} \cdot c_{\underline{d'}}^{-1}\theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee} = \begin{cases} 0 & \text{if } \mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d'})} \\ 0 & \text{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d'})} \text{ and } \underline{d} \not\sim_R \underline{c'} \\ c_{\underline{d'}}^{-1}\theta_{\underline{c''}}\theta_{\underline{d'}}^{\vee} & \text{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d'})} \text{ and } \underline{d} \sim_R \underline{c'} \end{cases}$$

Here, $\underline{c''}$ in the last case is the unique element with $\underline{c''} \sim_L \underline{d'}$ and $\underline{c''} \sim_R \underline{c}$ and the statement contains the information that such a $\underline{c''}$ in fact exists.

Proof. By Theorem 4.5 the elements $c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}$ and $c_{\underline{d'}}^{-1}\theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee}$ both lie in an isotypic component. Thus, if $\mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d'})}$ then clearly their product is zero.

Now assume that the left cell modules are isomorphic. Let Γ be an arbitrary left cell, such that $K\,\mathrm{LC}^{(\Gamma)}$ is isomorphic to $K\,\mathrm{LC}^{(\underline{d})}$ and $K\,\mathrm{LC}^{(\underline{d'})}$ and denote the corresponding irreducible character by χ. By Theorem 4.9 there are unique $\underline{a}, \underline{b}, \underline{a'}, \underline{b'} \in \Gamma$ with

$$\underline{a} \sim_R \underline{c} \quad\text{and}\quad \underline{b} \sim_R \underline{d} \quad\text{and}\quad \underline{a'} \sim_R \underline{c'} \quad\text{and}\quad \underline{b'} \sim_R \underline{d'}$$

and we have $\theta_{\underline{a}}\,\theta_{\underline{b}}^{\vee} = \theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}$ and $\theta_{\underline{a'}}\theta_{\underline{b'}}^{\vee} = \theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee}$. Thus, Theorem 4.5 implies that the product in the theorem is 0 if $\underline{b} \neq \underline{a'}$ and equal to $c_{\chi}^{-1}\theta_{\underline{a}}\,\theta_{\underline{b'}}^{\vee}$ otherwise. We remark that if $\underline{d} \sim_R \underline{c'}$, then $\underline{a'} \sim_R \underline{b}$ by transitivity. But using Proposition 3.6, $\underline{a'}, \underline{b} \in \Gamma$ implies $\underline{b} = \underline{a'}$. Hence $\underline{b} = \underline{a'}$ if and only if $\underline{d} \sim_R \underline{c'}$ which proves case two in the equation.

Finally, we assume also $\underline{d} \sim_R \underline{c'}$. Then, as $\underline{c''}$ runs through the left cell that contains $\underline{d'}$, we can apply Theorem 4.9 to each $\theta_{\underline{c''}}\theta_{\underline{d'}}^{\vee}$ and the left cell Γ. Since $\underline{b'} \in \Gamma$ and $\underline{b'} \sim_R \underline{d'}$ we get that

$$\{\theta_{\underline{c''}}\theta_{\underline{d'}}^{\vee} \mid \underline{c''} \sim_L \underline{d'}\} = \{\theta_{\underline{a''}}\theta_{\underline{b'}}^{\vee} \mid \underline{a''} \in \Gamma\}$$

and both sets have cardinality $|\Gamma|$. Thus, there is a unique $\underline{c''}$ with $\theta_{\underline{c''}}\theta_{\underline{d'}}^{\vee} = \theta_{\underline{a}}\,\theta_{\underline{b'}}^{\vee}$ characterised by $\underline{a} \sim_R \underline{c''} \sim_L \underline{d'}$ and the theorem is proved. □

Corollary 4.12 (Idempotents)**.** *The elements $c_{\underline{d}}^{-1}\theta_{\underline{d}}\,\theta_{\underline{d}}^{\vee}$ with $\underline{d} \in \mathcal{D}(n,r)$ are pairwise orthogonal primitive idempotents whose sum is the identity $1 \in \mathcal{S}_q(n,r)$. The central primitive idempotent corresponding to an irreducible character χ of $K\mathcal{S}_q(n,r)$ is equal to*

$$\sum_{\substack{\underline{d} \in \mathcal{D}(n,r) \\ \mathrm{LC}^{(\underline{d})} \text{ has character } \chi}} c_{\underline{d}}^{-1}\theta_{\underline{d}}\,\theta_{\underline{d}}^{\vee}$$

Proof. This follows directly from Theorems 4.11, 4.9 and 4.5. □

Corollary 4.13 (Left cell modules as submodules)**.** *Let $\underline{d} \in \mathcal{D}(n,r)$. Then the A-span*

$$\mathcal{L}_{\underline{d}} := \langle \theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee} \mid \underline{c} \sim_L \underline{d}\rangle_A$$

is a left $\mathcal{S}_q(n,r)$-module by the multiplication in $K\mathcal{S}_q(n,r)$ that is isomorphic to the left cell module $\mathrm{LC}^{(\underline{d})}$. In fact, the representing matrices with respect to the basis $(\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee})_{\underline{c}\sim_L\underline{d}}$

are equal to the representing matrices coming from the left cell module $\mathrm{LC}^{(\underline{d})}$ *with respect to its standard basis.*

Proof. Let Γ be the left cell that contains $\underline{d}$. Then by Formula (4.1) we have for every $h \in \mathcal{S}_q(n,r)$:

$$h\theta_{\underline{c}} = \sum_{\underline{c}' \in M(n,r)} \tau(\theta^\vee_{\underline{c}'} \cdot h\theta_{\underline{c}}) \cdot \theta_{\underline{c}'}.$$

Moreover, for $\underline{a} \in A$, there is $\alpha_{\underline{a}} \in A$ such that

$$h = \sum_{\underline{a} \in M(n,r)} \alpha_{\underline{a}} \theta_{\underline{a}}.$$

Hence, for $\underline{c}$, $\underline{c}' \in M(n,r)$, we have $\tau(\theta^\vee_{\underline{c}'} \cdot h\theta_{\underline{c}}) \in A$, because $\tau(\theta^\vee_{\underline{c}'} \cdot \theta_{\underline{a}}\theta_{\underline{c}}) \in A$ (see Remark 4.1). Multiplying this from the right with $\theta^\vee_{\underline{d}}$ we get

$$h\theta_{\underline{c}}\,\theta^\vee_{\underline{d}} = \sum_{\underline{c}' \in M(n,r)} \tau(h\theta_{\underline{c}}\,\theta^\vee_{\underline{c}'}) \cdot \theta_{\underline{c}'}\theta^\vee_{\underline{d}},$$

where we only have to sum over $\underline{c}' \in \Gamma$, since all the summands are zero unless $\underline{d} \leq_L \underline{c}' \leq_L \underline{c}$ by Lemma 4.2, which is equivalent to $\underline{c}' \in \Gamma$. We then deduce that $\mathcal{L}_{\underline{d}}$ is a left $\mathcal{S}_q(n,r)$-module. Moreover, comparing with Remark 4.1, this shows the statement about the representing matrices. □

Corollary 4.14. *The Schur algebra* $\mathcal{S}_q(n,r)$ *is contained in the* A*-span of the Wedderburn basis* $\mathcal{B}$*:*

$$\mathcal{S}_q(n,r) \subseteq \langle \mathcal{B} \rangle_A$$

Proof. Let $\Gamma_1, \ldots, \Gamma_n$ be left cells, such that the corresponding left cell modules form a system of representatives for the isomorphism types of simple left $K\mathcal{S}_q(n,r)$-modules. The mapping that maps $h \in K\mathcal{S}_q(n,r)$ to its tuple of representing matrices in the cell modules $\mathrm{LC}^{(\Gamma_1)}, \ldots, \mathrm{LC}^{(\Gamma_n)}$ with respect to their standard basis is an explicit isomorphism to a direct sum of full matrix rings over K. In this isomorphism, the elements of $\mathcal{B}$ are mapped to a matrix unit, that is, to tuples of matrices, in which exactly one matrix is non-zero, and this matrix contains exactly one non-zero coefficient equal to 1. The elements of $\mathcal{S}_q(n,r)$ are mapped to tuples of matrices with entries in A, since their representing matrices on the cell modules have entries in A (see the remark after Theorem 4.3). Therefore, $\mathcal{S}_q(n,r)$ lies in the A-span of $\mathcal{B}$. □

Proposition 4.15. *Let* τ *be a non-degenerate symmetrising trace form on* $K\mathcal{S}_q(n,r)$*. We denote by* $\mathcal{B}$ *the corresponding Wedderburn basis obtained in Theorem 4.11. Then, the dual basis of* $\mathcal{B}$ *relative to* τ *is*

$$\mathcal{B}^\vee = \{\theta_{\underline{c}}\theta^\vee_{\underline{d}} \mid \underline{c} \in M(n,r),\ \underline{d} \in \mathcal{D}(n,r),\ \underline{c} \sim_L \underline{d}\}.$$

Proof. Note first, that since τ is non-degenerate and $\mathcal{B}$ is a basis of $K\mathcal{S}_q(n,r)$, there must be at least one element $c^{-1}_{\underline{d}'}\,\theta_{\underline{c}'}\theta^\vee_{\underline{d}'} \in \mathcal{B}$ such that $\tau(c^{-1}_{\underline{d}}\,\theta_{\underline{c}}\,\theta^\vee_{\underline{d}} \cdot c^{-1}_{\underline{d}'}\,\theta_{\underline{c}'}\theta^\vee_{\underline{d}'})$ is non-zero. Since $c_{\underline{d}'} \neq 0$, we have in particular $\tau(c^{-1}_{\underline{d}}\,\theta_{\underline{c}}\,\theta^\vee_{\underline{d}}\theta_{\underline{c}'}\theta^\vee_{\underline{d}'}) \neq 0$. We try to find out, which element $\theta_{\underline{c}'}\theta^\vee_{\underline{d}'}$ this can be:

By Theorem 4.11, the value $\tau(c^{-1}_{\underline{d}}\,\theta_{\underline{c}}\,\theta^\vee_{\underline{d}}\,\theta_{\underline{c}'}\theta^\vee_{\underline{d}'})$ is equal to zero, if $\mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d}')}$ or $\underline{d} \not\sim_R \underline{c}'$. If however $\mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}')}$ and $\underline{d} \sim_R \underline{c}'$, then it is equal to $\tau(\theta_{\underline{c}''}\theta^\vee_{\underline{d}'})$ where $\underline{c}''$ is uniquely defined by $\underline{c}'' \sim_L \underline{d}'$ and $\underline{c}'' \sim_R \underline{c}$. If $\underline{c}'' \neq \underline{d}'$, then this value is also equal to 0 because of the original definition of $\{\theta^\vee_{\underline{a}} \mid \underline{a} \in M(n,r)\}$. If however $\underline{c}'' = \underline{d}'$ we can show that $\underline{c'} = \underline{c}^t$ using Proposition 3.6: Namely, we have $\underline{c'} \sim_L \underline{d'} = \underline{c''} \sim_R \underline{c}$ and thus $\underline{c'} \sim_L \underline{c}^t$ by transposition. Further, we have $\underline{c'} \sim_R \underline{d} \sim_L \underline{c}$ and thus again by transposition $\underline{c'} \sim_R \underline{c}^t$. Thus, $\underline{c'}$ and $\underline{c}^t$ are both left and right equivalent and therefore equal.

Thus, we deduce that

$$\tau(c_{\underline{d}}^{-1}\, \theta_{\underline{c}}\, \theta_{\underline{d}}^{\vee} \cdot \theta_{\underline{c'}} \theta_{\underline{d'}}^{\vee}) = \delta_{\underline{c'},\underline{c}^t}$$

for all $\underline{c} \in M(n,r)$ and $\underline{d} \in \mathcal{D}(n,r)$ with $\underline{c} \sim_L \underline{d}$, and all $\underline{c'} \in M(n,r)$ and $\underline{d'} \in \mathcal{D}(n,r)$ with $\underline{c'} \sim_L \underline{d'}$. □

Remark 4.16. Note that as a byproduct we have proved the following result: If $\underline{c} \in M(n,r)$ and $\underline{d} \in \mathcal{D}(n,r)$ with $\underline{c} \sim_L \underline{d}$, and $\underline{d'} \in \mathcal{D}(n,r)$ with $\underline{c}^t \sim_L \underline{d'}$, then $\mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d'})}$.

We now talk about A-sublattices of $K\mathcal{S}_q(n,r)$.

Definition/Proposition 4.17 (A-sublattices of $K\mathcal{S}_q(n,r)$ and their duals)**.** By an **A-lattice in** $K\mathcal{S}_q(n,r)$ we mean an A-free A-submodule that contains a K-basis of $K\mathcal{S}_q(n,r)$. Let $L \subseteq K\mathcal{S}_q(n,r)$ be an A-lattice. Then we set

$$L^{\vee} := \{h \in K\mathcal{S}_q(n,r) \mid \tau(hx) \in A \text{ for all } x \in L\}$$

and call it the **dual lattice of** L. Since τ is non-degenerate, $L^{\vee}$ is again an A-lattice in $K\mathcal{S}_q(n,r)$, namely, if $(b_{\underline{a}})_{\underline{a} \in M(n,r)}$ is an A-basis of L, then the dual basis $(b_{\underline{a}}^{\vee})_{\underline{a} \in M(n,r)}$ is an A-basis of $L^{\vee}$. Clearly, if $L \subseteq N$ are two A-lattices in $K\mathcal{S}_q(n,r)$, then $N^{\vee} \subseteq L^{\vee}$.

Note that we do not require an A-lattice to be an A-algebra! □

Proposition 4.18 (The dual is an $\mathcal{S}_q(n,r)$-module)**.** *We have* $\mathcal{S}_q(n,r) \cdot \mathcal{S}_q(n,r)^{\vee} \subseteq \mathcal{S}_q(n,r)^{\vee}$.

Proof. Fix $h \in \mathcal{S}_q(n,r)$ and $k \in \mathcal{S}_q(n,r)^{\vee}$. We have to show that $hk \in \mathcal{S}_q(n,r)^{\vee}$. However, for every $x \in \mathcal{S}_q(n,r)$ holds $\tau(hkx) = \tau(kxh)$. Since $xh \in \mathcal{S}_q(n,r)$ (because $\mathcal{S}_q(n,r)$ is an algebra), and $k \in \mathcal{S}_q(n,r)^{\vee}$ we get $\tau(kxh) \in A$. □

For the rest of this section we let $\tau = \sum_{\chi \in \mathrm{Irr}(K\mathcal{S}_q(n,r))} \chi$, that is, we choose τ such that all Schur elements are equal to 1.

Proposition 4.19 (The Wedderburn-basis is self-dual)**.** *Let* $\tau = \sum_{\chi \in \mathrm{Irr}(K\mathcal{S}_q(n,r))} \chi$*. Then*

$$\langle \mathcal{B} \rangle_A^{\vee} = \langle \mathcal{B} \rangle_A$$

for the Wedderburn basis $\mathcal{B}$ *from Theorem 4.11.*

Proof. Since τ is the sum of the irreducible characters, all Schur elements c_χ are equal to one. It is then a direct consequence of Proposition 4.15. □

Corollary 4.20 (The dual of $\mathcal{S}_q(n,r)$)**.** *From Lemma 4.14 and Proposition 4.19 follows*

$$\langle \mathcal{B} \rangle_A \subseteq \mathcal{S}_q(n,r)^{\vee}$$

Proof. Dualising reverses inclusion. □

5. The asymptotic algebra and the Du-Lusztig homomorphism

In this section we briefly recall the definition of the asymptotic algebra $\mathcal{J}(n,r)$ for the q-Schur algebra $\mathcal{S}_q(n,r)$ and of the Du-Lusztig homomorphism Φ from $\mathcal{S}_q(n,r)$ to $\mathcal{J}(n,r)$. We then show that this algebra is isomorphic to the algebra $\langle \mathcal{B} \rangle_A$ spanned by our Wedderburn basis $\mathcal{B}$ and that the Du-Lusztig homomorphism can be interpreted as the inclusion of $\mathcal{S}_q(n,r)$ into $\langle \mathcal{B} \rangle_A$.

Definition 5.1 (The asymptotic algebra $\mathcal{J}(n,r)$)**.** Let $\mathcal{J}(n,r)$ be the free abelian group with basis $\{t_{\underline{a}} \mid \underline{a} \in M(n,r)\}$. We define a multiplication on $\mathcal{J}(n,r)$ by setting

$$t_{\underline{a}} t_{\underline{b}} = \sum_{\underline{c} \in M(n,r)} \gamma_{\underline{a},\underline{b},\underline{c}^t} \cdot t_{\underline{c}}.$$

We set $\mathcal{D}(n,r)_\lambda := \mathcal{D}(n,r) \cap M_{\lambda,\lambda}$. Following Du, we denote the extension of scalars of $\mathcal{J}(n,r)$ to A by $\mathcal{J}(n,r)_A$.

Lemma 5.2 (See [7, (2.2.1)])**.** *The $\mathbb{Z}$-algebra $\mathcal{J}(n,r)$ is associative with the identity element*

$$\sum_{\underline{d} \in \mathcal{D}(n,r)} t_{\underline{d}}.$$

Theorem 5.3 (The Du-Lusztig homomorphism Φ, see [7, (2.3])**.** *The A-linear map Φ : $\mathcal{S}_q(n,r) \to \mathcal{J}(n,r)_A$ defined by*

$$\Phi(\theta_{\underline{a}}) := \sum_{\substack{\underline{b} \in M(n,r) \\ \underline{d} \in \mathcal{D}(n,r)_\mu \\ \mathbf{a}(\underline{d}) = \mathbf{a}(\underline{b})}} f_{\underline{a},\underline{d},\underline{b}} \cdot t_{\underline{b}} = \sum_{\substack{\underline{b} \in M(n,r) \\ \underline{d} \in \mathcal{D}(n,r) \\ \underline{d} \sim_L \underline{b}}} f_{\underline{a},\underline{d},\underline{b}} \cdot t_{\underline{b}}, \qquad \text{where } \mu = co(\underline{a})$$

is an algebra homomorphism and becomes an isomorphism $K\mathcal{S}_q(n,r) \to \mathcal{J}(n,r)_K$ when tensored with the field of fractions K of A.

Proof. See [7, 2.3]. The latter equation holds, since $f_{\underline{a},\underline{b},\underline{d}} = 0$ unless $\underline{d} \leq_L \underline{b}$, and **Q9** implies $\underline{d} \sim_L \underline{b}$ in this case. Also we can safely sum over all of $\mathcal{D}(n,r)$ neglecting the index μ, since all elements $\underline{d} \in \mathcal{D}(n,r)$ fulfill $ro(\underline{d}) = co(\underline{d})$ by definition (see Definition 2.5 and the remark there) and $f_{\underline{a},\underline{d},\underline{b}} = 0$ unless $co(\underline{a}) = ro(\underline{d})$ anyway. □

We can now present our main theorem, which links our Wedderburn basis $\mathcal{B}$ to the asymptotic algebra:

Theorem 5.4 (Preimage of the t-basis under the Du-Lusztig homomorphism)**.** *Let τ be an arbitrary non-degenerate symmetrising trace form. All dual bases in the following are meant with respect to τ.*

With the above notation we have

$$\Phi(c_{\underline{d}}^{-1} \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee) = t_{\underline{c}} \qquad \text{for all } \underline{c} \in M(n,r).$$

Proof. The rightmost sum in Theorem 5.3 has the advantage that it provides a formula for the image of an arbitrary element $h \in K\mathcal{S}_q(n,r)$ under the Du-Lusztig homomorphism, since it is obviously K-linear in $\theta_{\underline{a}}$:

$$\Phi(h) = \sum_{\substack{\underline{b} \in M(n,r) \\ \underline{d}' \in \mathcal{D}(n,r) \\ \underline{d}' \sim_L \underline{b}}} \tau(h \cdot \theta_{\underline{d}'} \theta_{\underline{b}}^\vee) \cdot t_{\underline{b}}$$

(recall $\tau(\theta_{\underline{a}} \theta_{\underline{d}'} \theta_{\underline{b}}^\vee) = f_{\underline{a},\underline{d}',\underline{b}}$). But now we can immediately set $h := c_{\underline{d}}^{-1} \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee$ for some $\underline{c} \in M(n,r)$ and $\underline{d} \in \mathcal{D}(n,r)$ with $\underline{c} \sim_L \underline{d}$. The value $\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee \cdot \theta_{\underline{d}'} \theta_{\underline{b}}^\vee)$ is zero (see Lemma 4.2) unless $\underline{b} \leq_R \underline{c} \sim_L \underline{d} \leq_R \underline{d}' \sim_L \underline{b}$ and this implies $\underline{b} \sim_R c$ and $\underline{d}' \sim_R \underline{d}$ using **Q4** and **Q10**. But this means $\underline{d}' = \underline{d}$ by **Q13** and the definition of $\sim_R$ and thus $\underline{b} = \underline{c}$ because of Lemma 3.6. Thus, in the sum there is only one non-zero summand, which is $\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee \cdot \theta_{\underline{d}}\, \theta_{\underline{c}}^\vee) t_{\underline{c}}$. Now everything is in a single left cell such that we can use Theorem 4.5 to get

$$\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee \cdot \theta_{\underline{d}}\, \theta_{\underline{c}}^\vee) \cdot t_{\underline{c}} = \tau(\theta_{\underline{c}}\, \theta_{\underline{c}}^\vee) \cdot t_{\underline{c}} = t_{\underline{c}}$$

as claimed. □

We can summarise our results in the following way:

Theorem 5.5 (New interpretation of the Du-Lusztig homomorphism)**.** *Let τ be an arbitrary non-degenerate symmetrising trace form on $K\mathcal{S}_q(n,r)$. We define the set $\mathcal{B}$ as in Theorem 4.11 and we set*

$$\mathcal{J}_\tau = \langle \mathcal{B} \rangle_A .$$

The following diagram commutes and all unmarked arrows are identities or natural inclusions:

$$\begin{array}{ccccc} \mathcal{S}_q(n,r) & \longrightarrow & \mathcal{J}_\tau & \longrightarrow & K\mathcal{S}_q(n,r) \\ \Big\| & & \Phi \Big\downarrow \cong & & \Phi \Big\downarrow \cong \\ \mathcal{S}_q(n,r) & \xrightarrow{\Phi} & \mathcal{J}(n,r)_A & \longrightarrow & \mathcal{J}(n,r)_K \end{array}$$

Thus, the asymptotic algebra $\mathcal{J}(n,r)_A$ is nothing but the A-span of our Wedderburn basis and the Du-Lusztig homomorphism Φ can simply be interpreted as the inclusion of $\mathcal{S}_q(n,r)$ into $\langle \mathcal{B} \rangle_A$. Furthermore, our results directly and explicitly show that $\langle \mathcal{B} \rangle_A$ is isomorphic as an A-algebra to a direct sum of full matrix rings over A.

6. A CRITERION FOR JAMES' CONJECTURE

In this section we show how our results provide an equivalent formulation of a conjecture about the representation theory of specialisations of the q-Schur algebra. We first recall the conjecture.

The construction of the Iwahori-Hecke algebra of type A and of the q-Schur algebra as in Section 2 together with their Kazhdan-Lusztig bases can be carried out over an arbitrary integral domain R with quotient field k and with an arbitrary invertible parameter $q \in R$ having a square root in that domain. We denote the resulting algebra by $\mathcal{S}_q(n,r)_R$ and its extension of scalars to k by $\mathcal{S}_q(n,r)_k$.

The case of the Laurent polynomial ring $A = \mathbb{Z}[v, v^{-1}]$ and $q = v^2$ is called the "generic" case, since for every other choice (R, q) there is a ring homomorphism $\varphi : \mathbb{Z}[v, v^{-1}] \to R$ mapping v^2 to $q \in R$, which induces a ring homomorphism $\mathcal{S}_{v^2}(n,r)_A \to \mathcal{S}_q(n,r)_R \subseteq \mathcal{S}_q(n,r)_k$. This is called a "specialisation".

It is known, that $\mathcal{S}_q(n,r)_k$ is semisimple unless q is an e-th root of unity. If q is a root of unity, then there is a decomposition matrix, which records the multiplicities of the simple modules in the so-called "standard modules". For the case that k has characteristic zero, recent work by Lascoux, Leclerc and Thibon, and Varagnolo and Vasserot yields a complete determination of these decomposition matrices (see [15], [8] and the references there). However, the case of positive characteristic is still open.

James' conjecture is a statement about this modular case. Roughly speaking, it asserts that if k is a field of characteristic ℓ and the multiplicative order e of the parameter $q \in k$ is greater than r, then the decomposition matrix of $\mathcal{S}_q(n,r)_k$ does not depend on the particular value of ℓ but only on e.

We now want to make this statement more precise. Both the simple modules and the standard modules have a labelling by the set $\Lambda(n,r)$. Let $V_{k,q}^\lambda$ denote the standard module and $M_{k,q}^\mu$ the simple module of $\mathcal{S}_q(n,r)_k$ corresponding to λ and μ respectively. Then the decomposition matrix for $\mathcal{S}_q(n,r)_k$ consists of the numbers

$$d_{\lambda,\mu}^{k,q} := \text{multiplicity of } M_{k,q}^\mu \text{ in } V_{k,q}^\lambda.$$

Conjecture 6.1 (James, see [11, §4] and [8, §3])**.** *If $\ell > r$ and e is the multiplicative order of $q \in k$, then $d_{\lambda,\mu}^{k,q} = d_{\lambda,\mu}^{\mathbb{Q}(\zeta_e),\zeta_e}$ for all $\lambda, \mu \in \Lambda(n,r)$, where ζ_e is a complex primitive e-th root of unity.*

Meinolf Geck has shown in [9, Theorem 1.2] that this statement is equivalent to the fact that, for $\ell > r$, the rank of the Du-Lusztig homomorphism $\Phi : \mathcal{S}_q(n,r)_k \to \mathcal{J}(n,r)_k$ with respect to the two bases $(\theta_{\underline{a}})_{\underline{a}\in M(n,r)}$ and $(t_{\underline{a}})_{\underline{a}\in M(n,r)}$ is equal to the rank of the corresponding Du-Lusztig homomorphism $\mathcal{S}_{\zeta_{2e}^2}(n,r)_{\mathbb{Q}(\zeta_{2e})} \to \mathcal{J}(n,r)_{\mathbb{Q}(\zeta_{2e})}$ with respect to the corresponding bases, where e is the multiplicative order of $q \in k$ and ζ_{2e} is a primitive $2e$-th root of unity in $\mathbb{C}$. In particular, the rank does not depend on the characteristic ℓ of k.

In view of our Theorem 5.5 this immediately implies:

Theorem 6.2 (An equivalent formulation of James' conjecture)**.** *Let $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ be the Du-Kazhdan-Lusztig-basis of $\mathcal{S}_q(n,r)$ and let τ be a non degenerate symmetrising trace form for $K\mathcal{S}_q(n,r)$. Let $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n,r)\}$ be the dual basis of $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ with respect to τ. Let $\mathcal{B}$ be the basis defined in Theorem 4.11. Let $s := |M(n,r)|$ and $M = (m_{\underline{a},\underline{b}})_{\underline{a},\underline{b}\in M(n,r)} \in A^{s\times s}$ be the matrix, for which*

$$\theta_{\underline{a}} = \sum_{\underline{c}\in M(n,r)} m_{\underline{a},\underline{c}} \cdot c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^\vee$$

with $c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^\vee \in \mathcal{B}$ holds for all $\underline{a} \in M(n,r)$.

Let ℓ be a prime and $\varphi_\ell : \mathbb{Z}[v,v^{-1}] \to \mathbb{F}_\ell$ a ring homomorphism, such that the multiplicative order of $\varphi_\ell(v)$ is equal to $2e$. Denote by $\varphi_\ell(M)$ the matrix in $\mathbb{F}_\ell^{s\times s}$ that one gets by applying the ring homomorphism φ_ℓ to every entry of M.

Let ζ_{2e} be a primitive $2e$-th root of unity in $\mathbb{C}$ and $\varphi_e : \mathbb{Z}[v,v^{-1}] \to \mathbb{Q}(\zeta_{2e})$ be the ring homomorphism mapping v to ζ_{2e}. Then there is a ring homomorphism $\varphi_\ell^e : \mathbb{Z}[\zeta_{2e}] \to \mathbb{F}_\ell$ with $\varphi_\ell = \varphi_\ell^e \circ \varphi_e$. Denote by $\varphi_e(M)$ the matrix in $\mathbb{Q}(\zeta_{2e})^{s\times s}$ that one gets by applying the ring homomorphism φ_e to every entry of M.

Then James' conjecture is equivalent to the fact that for $\ell > r$ the ranks of $\varphi(M)$ (over $\mathbb{F}_\ell$) and of $\varphi_e(M)$ (over $\mathbb{Q}(\zeta_{2e})$) are equal.

Let τ be a non-degenerate symmetrising trace form on $K\mathcal{S}_q(n,r)$. We denote by $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ the Du-Kazhdan-Lusztig-basis of $\mathcal{S}_q(n,r)$ and by $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n,r)\}$ its dual basis relative to τ. As above, we denote by $\mathcal{B}$ the Wedderburn basis obtained in Theorem 4.11. Moreover, we denote by $M = (m_{\underline{a},\underline{b}})_{\underline{a},\underline{b}\in M(n,r)}$ the change of basis matrix from $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ to $\mathcal{B}$ as above and by $P_\tau = (p_{\underline{a},\underline{b}})_{\underline{a},\underline{b}\in M(n,r)}$ the change of basis matrix from $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ to $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n,r)\}$, that is:

$$\theta_{\underline{a}} = \sum_{\underline{b}\in M(n,r)} p_{\underline{a},\underline{b}} \cdot \theta_{\underline{b}}^\vee$$

for all $\underline{a} \in M(n,r)$. Formula (4.1) implies that

$$P_\tau = \left(\tau(\theta_{\underline{a}}\theta_{\underline{b}})\right)_{\underline{a},\underline{b}\in M(n,r)} \quad \text{and} \quad P_\tau^{-1} = \left(\tau(\theta_{\underline{a}}^\vee\theta_{\underline{b}}^\vee)\right)_{\underline{a},\underline{b}\in M(n,r)}.$$

Lemma 6.3. *With the above notation, the matrix*

$$D = M^T P_\tau^{-1} M$$

is monomial and its entries are the Schur elements $c_{\underline{d}}$ associated to $\underline{d} \in \mathcal{D}(n,r)$ as in Definition 4.10.

Proof. The matrix M^T is the change of basis matrix from $\mathcal{B}^\vee$ to $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n,r)\}$ and thus the matrix D is the change of basis matrix from $\mathcal{B}^\vee$ to $\mathcal{B}$, that is:

$$\theta_{\underline{c}}\theta_{\underline{d}}^\vee = \sum_{\underline{c'} \in M(n,r)} d_{\underline{c},\underline{c'}} c_{\underline{d'}}^{-1} \theta_{\underline{c'}} \theta_{\underline{d'}}^\vee$$

for all $\theta_{\underline{c}}\theta_{\underline{d}}^\vee \in \mathcal{B}^\vee$. Using Proposition 4.15, the result follows. □

Proposition 6.4 (A criterion for James' conjecture)**.** *Let τ be a non-degenerate symmetrising trace form on $K\mathcal{S}_q(n,r)$. Let $\varphi_e : A \to \mathbb{Z}[\zeta_{2e}], v \mapsto \zeta_{2e}$ be a specialisation to characteristic 0 where v^2 is mapped to a primitive e-th root of unity in a cyclotomic field and $\varphi_\ell : A \to \mathbb{F}_\ell$ is a second specialisation to characteristic ℓ such that there is a ring homomorphism $\varphi_\ell^e : \mathbb{Z}[\zeta_{2e}] \to \mathbb{F}_\ell$ with $\varphi_\ell = \varphi_\ell^e \circ \varphi_e$. We suppose that $\ell > r$ and the following hypotheses on τ:*

- *The Schur elements $c_{\underline{d}}$ for $\underline{d} \in \mathcal{D}(n,r)$ lie in A.*
- *The coefficients of the matrix P_τ^{-1} lie in A.*
- *Let a be the number of Schur elements $c_{\underline{d}}$ for $\underline{d} \in \mathcal{D}(n,r)$ that do not vanish under φ_e and b the number of Schur elements that do not vanish under φ_ℓ. The numbers a and b are both equal to the rank over $\mathbb{Q}(\zeta_{2e})$ of the matrix $\varphi_e(M)$ for M from above.*

Note that we denote with the notation $\varphi_e(M)$ the matrix one gets from M by applying the ring homomorphism φ_e on every entry.

If τ can be found fulfilling all these hypotheses, then James' conjecture holds for all $\ell > r$ for which φ_ℓ as above exist.

Proof. We denote by M the change of basis matrix from $\{\theta_{\underline{a}} \mid \underline{a} \in M(n,r)\}$ to $\mathcal{B}$ as above. Then Lemma 6.3 asserts that

$$D = M^T P_\tau^{-1} M.$$

Thanks to Theorem 4.11, the coefficients of the matrix M lie in A. By hypothesis, the matrix P_τ^{-1} has coefficients in A. By Lemma 6.3 and the first hypothesis the entries of D are also in A.

Since the matrices D, M, M^T, and P_τ^{-1} have coefficients in A, the matrices $\varphi_e(D)$, $\varphi_e(M)$, $\varphi_\ell(D)$, $\varphi_\ell(M)$, $\varphi_\ell(M^T)$ and $\varphi_\ell(P_\tau^{-1})$ are well-defined. We then have the following equality

$$\varphi_\ell(D) = \varphi_\ell(M^T) \cdot \varphi_\ell(P_\tau^{-1}) \cdot \varphi_\ell(M),$$

implying that $\mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(D)) \leq \mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M))$. Moreover we have $\varphi_\ell(M) = \varphi_\ell^e(\varphi_e(M))$. Since φ_ℓ^e is a ring homomorphism, we deduce that

$$\mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M)) \leq \mathrm{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_e(M)).$$

Since D is a monomial matrix containing only the Schur elements as non-zero entries, the numbers a and b from the hypotheses are the ranks of $\varphi_e(D)$ and $\varphi_\ell(D)$ respectively. However, if as in the last hypothesis the ranks of $\varphi_e(M)$ and $\varphi_\ell(D)$ are equal, then it follows that $\mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M)) \leq \mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(D))$. We then deduce that

$$\mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M)) = \mathrm{rk}_{\mathbb{F}_\ell}(\varphi_\ell(D)),$$

and the result now follows from Theorem 6.2. □

Remark 6.5. To prove James' conjecture it is enough to find a symmetrising trace form τ on $K\mathcal{S}_q(n,r)$ such that the hypotheses of Proposition 6.4 are satisfied. We notice that the assumption on P_τ in the statement of Proposition 6.4 is "generic" in the sense that this property only depending on the "generic" q-Schur algebra, but not on specialisations over finite fields.

Remark 6.6. We can replace the second assumption of Proposition 6.4 by the fact that the matrix $P_\tau^{-1}M$ (or $M^T P_\tau^{-1}$) has its coefficients in A.

Remark 6.7. For the usual trace form τ on Iwahori-Hecke algebras of type A, we note that the assumptions of Proposition 6.4 hold. Then using [14], we can prove in a way similar to the one of the proof of Proposition 6.4 that the rank of the Lusztig homomorphim (specialized in a finite field $\mathbb{F}_\ell$ by $\varphi_\ell : A \to \mathbb{F}_\ell$ mapping v^2 to an element $q \in \mathbb{F}_\ell$ with multiplicative order e as above) does not depend on ℓ. However as noted by Geck in [9] an analogue result as Theorem 6.2 in Iwahori-Hecke algebras does not imply the Iwahori-Hecke algebras James' conjecture.

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