

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Sudoku and Mathematics

Max Neunhöffer



University of St Andrews

Edinburgh 21 January 2011

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Equivalent Sudokus

Symmetry Breaking

## Open problems

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Backtrack search

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Equivalent Sudokus

Symmetry Breaking

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## Rule

Each row, column and  $3 \times 3$ -block contains the numbers 1 to 9 **each exactly once**.

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## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

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What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

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## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

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The Hitting Set Problem

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What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

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## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

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What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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If there is a **unique solution**, this method will find it.

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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⇒ Solving Sudokus is mathematics!

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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$\implies$  Solving Sudokus is **not** a difficult problem in **CompSci**.

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There is a **whole branch** of CompSci/Maths called **Constraint Satisfaction**: deals with **very hard problems similar to Solving Sudokus**

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What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Number of $9 \times 9$ latin squares

There are altogether

$$5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \approx 5.525 \cdot 10^{27}$$

**latin squares** of size  $9 \times 9$ .

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How could one possibly count these?

**Not by brute force**: If our computer enumerates **one** in **each clock cycle**, it would need **90000 years** to finish!

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Counting full Sudoku grids

Consider only the first block row:

- 1 We can **renumber** to get this left hand  $3 \times 3$ -block:

1	2	3	?	?	?	?	?	?
4	5	6	?	?	?	?	?	?
7	8	9	?	?	?	?	?	?

Count those, multiply by  $9! = 362880$  in the end.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Count those, multiply by  $9! = 362880$  in the end.

- 2 Distinguish more cases and **identify different cases**, which have the same number of completions.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- 3 Finally, run a (backtrack) computer search for each case to **count the possibilities**.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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7	8	9	?	?	?	?	?	?

Count those, multiply by  $9! = 362880$  in the end.

- 2 Distinguish more cases and **identify different cases**, which have the same number of completions.
- 3 Finally, run a (backtrack) computer search for each case to **count the possibilities**.

## The answer

There are altogether

$$6\,670\,903\,752\,021\,072\,936\,960 \approx 6.671 \cdot 10^{21}$$

different full Sudoku grids. (Felgenhauer/Jarvis 2006)

<http://www.afjarvis.staff.shef.ac.uk/sudoku/>

# Equivalence of Sudokus

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

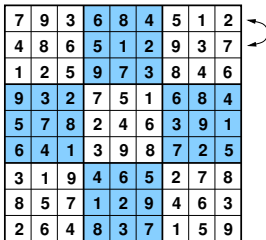
How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

7	9	3	6	8	4	5	1	2
4	8	6	5	1	2	9	3	7
1	2	5	9	7	3	8	4	6
9	3	2	7	5	1	6	8	4
5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
8	5	7	1	2	9	4	6	3
2	6	4	8	3	7	1	5	9



Equivalence transformations:

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

## Equivalence of Sudokus

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5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
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## Equivalence transformations:

- **Permute:** rows in a block,

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

## Equivalence of Sudokus

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1	2	5	9	7	3	8	4	6
9	3	2	7	5	1	6	8	4
5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
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## Equivalence transformations:

- **Permute**: rows in a block, columns in a block,

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

## Equivalence of Sudokus

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5	7	8	2	4	6	3	9	1
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## Equivalence transformations:

- **Permute**: rows in a block, columns in a block, block-rows,

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

## Equivalence of Sudokus

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## Equivalence transformations:

- **Permute**: rows in a block, columns in a block, block-rows, block-columns

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Equivalence of Sudokus

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## Equivalence transformations:

- **Permute**: rows in a block, columns in a block, block-rows, block-columns
- **Renumber**: entries

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

## Equivalence of Sudokus

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2	6	4	8	3	7	1	5	9

## Equivalence transformations:

- **Permute**: rows in a block, columns in a block, block-rows, block-columns
- **Renumber**: entries
- **Flip**: entire grid

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

## Equivalence of Sudokus

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2	6	4	8	3	7	1	5	9

## Equivalence transformations:

- **Permute**: rows in a block, columns in a block, block-rows, block-columns
  - **Renumber**: entries
  - **Flip**: entire grid
- All concatenations of these form a **group**.

Sudoku

What is a Sudoku?

Solving Sudoku

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudoku

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Equivalence of Sudokus

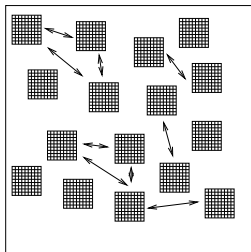
## Definition: Equivalent Sudokus

Two Sudoku grids/puzzles are called **equivalent** if one arises from the other by applying a sequence of equivalence transformations.

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Sudoku

What is a Sudoku?

Solving Sudoku

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

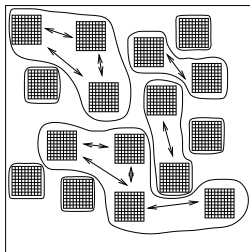
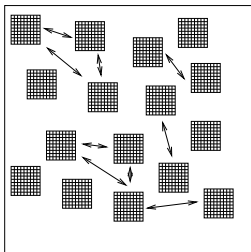
The Hitting Set Problem

Backtrack Search

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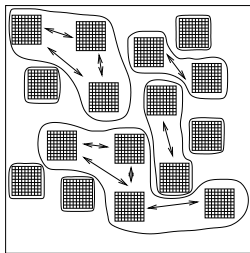
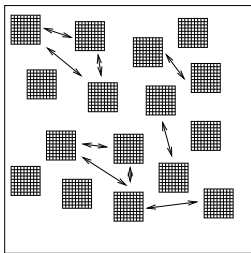


We form **equivalence classes** or **orbits**.

# Equivalence of Sudokus

## Definition: Equivalent Sudokus

Two Sudoku grids/puzzles are called **equivalent** if one arises from the other by applying a sequence of **equivalence transformations**.



We form **equivalence classes** or **orbits**.

→ There are **5 472 730 538** classes (Jarvis/Russell 2006)

<http://www.afjarvis.staff.shef.ac.uk/sudoku/>

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Symmetry Breaking

We “break the symmetry” by considering **exactly one from each equivalence class.**

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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① We can **renumber** to get this left hand  $3 \times 3$ -block:

1	2	3	4	6	8	9	5	7
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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- 2 Distinguish cases for first row:

(Type I)

1	2	3	{4,5,6}	{7,8,9}
4	5	6		
7	8	9		

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- ② Distinguish cases for first row:

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1	2	3	{4,5,6}	{7,8,9}
4	5	6	{7,8,9}	{1,2,3}
7	8	9	{1,2,3}	{4,5,6}

(Type II)

1	2	3	{4,5,7}	{6,8,9}
4	5	6		
7	8	9		

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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7	8	9	{1,2,3}	{4,5,6}

(Type II)

1	2	3	{4,5,7}	{6,8,9}
4	5	6	{8,9,a}	{7,b,c}
7	8	9	{6,b,c}	{4,5,a}

where  $\{a, b, c\} = \{1, 2, 3\}$ .

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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(Type II)

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where  $\{a, b, c\} = \{1, 2, 3\}$ .

- ③ Some more such arguments ...

# Open problems about Sudoku

- How many **Sudoku-puzzles** are there?

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

### 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Open problems about Sudoku

- How many **Sudoku-puzzles** are there?
- How many **essentially different Sudoku-puzzles** are there?

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

### 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Open problems about Sudoku

- How many **Sudoku-puzzles** are there?
- How many **essentially different Sudoku-puzzles** are there?
- How many **Sudoku-puzzles in which no clue can be left out** are there?

Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Open problems about Sudoku

- How many **Sudoku-puzzles** are there?
- How many **essentially different** **Sudoku-puzzles** are there?
- How many **Sudoku-puzzles in which no clue can be left out** are there? (They are called **irreducible**.)

Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- Given a conclusion strategy, **how many** **Sudoku-puzzles** can be solved with only conclusions?

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- Given a conclusion strategy, **how many** **Sudoku-puzzles** can be solved with only conclusions?
- What is the **maximum number of clues** in an **irreducible Sudoku-puzzle**?

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- Given a conclusion strategy, **how many** **Sudoku-puzzles** can be solved with only conclusions?
- What is the **maximum number of clues** in an **irreducible Sudoku-puzzle**?
- What is the **minimum number of clues** in a **Sudoku-puzzle**?

Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- How many **essentially different** **Sudoku-puzzles** are there?
- How many **Sudoku-puzzles in which no clue can be left out** are there? (They are called **irreducible**.)
- Given a conclusion strategy, **how many** **Sudoku-puzzles** can be solved with only conclusions?
- What is the **maximum number of clues** in an **irreducible Sudoku-puzzle**?
- What is the **minimum number of clues** in a **Sudoku-puzzle**?
- We do not even have started to consider  **$16 \times 16$ -Sudokus**...

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# What is the smallest number of clues?

Of course: such that there is a **unique solution**.

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# What is the smallest number of clues?

Of course: such that there is a **unique solution**.

- **Gordon Royle** (University of Western Australia, Perth) maintains a collection of currently **49151 pairwise inequivalent** Sudoku puzzles with 17 clues.

<http://www.csse.uwa.edu.au/~gordon/sudokumin.php>

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- He **reckons** “that new 17-clue Sudoku puzzles are becoming **rarer** to find”.
- There is a set of 16 clues which allows **exactly two solutions**.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

							1	
4								
	2							
				5		6		4
		8				3		
		1		9				
3			4			2		
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This puzzle has 17 clues.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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This puzzle has 17 clues. **None of them** can be left out.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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	5		1				
			8		7		

This puzzle has 17 clues. **None of them** can be left out.

**Question:**

Are there 16 clues which uniquely define a Sudoku grid?

**This is an unsolved problem!**

## Question

Fix one Sudoku grid. Can it be the solution to a 16-clue Sudoku puzzle?

### Sudoku

What is a Sudoku?

### Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

### Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

### Open problems

#### 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Question

Fix one Sudoku grid. Can it be the solution to a 16-clue Sudoku puzzle?

Number of ways to choose 16 out of 81:

$$\binom{81}{16} = 33\,594\,090\,947\,249\,085 \approx 33 \cdot 10^{15}$$

Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

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Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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**Idea:** We do not have to try all choices.

We need **constraints** that the selection of 16 has to fulfill.

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## Definition: Unavoidable Set

Let  $S$  be a filled Sudoku grid. A subset  $U$  of the 81 positions is called an **unavoidable set**, if **every** set of clues **uniquely defining  $S$**  has a number in at least one of the positions in  $U$ .

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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- Are there unavoidable sets and if so how many?
- How can we find them?
- How does this help?

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

7	9	3	6	8	4	5	1	2
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1	2	5	9	7	3	8	4	6
9	3	2	7	5	1	6	8	4
5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
8	5	7	1	2	9	4	6	3
2	6	4	8	3	7	1	5	9

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Any set of 16 clues cannot avoid the yellow positions.

## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

7	9	3	6	8	4	5	1	2
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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

7	9	3	6	8	4	5	1	2
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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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5	7	8	2	4	6		9	
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3				6		2	7	8
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## Sudoku

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Any set of 16 clues cannot avoid the yellow positions.  
Because this puzzle has **more than one solution**.

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# The Hitting Set Problem

## Problem: Hitting Set (resp. Set Covering)

Let  $M$  be a set and let  $A_1, \dots, A_k$  be subsets of  $M$ .

Find a **minimal** subset  $H$  of  $M$  which contains at **least one element of every  $A_i$**  for  $1 \leq i \leq k$ .

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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This problem is **computationally hard**.

Sudoku

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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It is one of Karp's 21 **NP-complete** problems ( $\rightarrow$  [1]).

[1] Richard M. Karp, *Reducibility Among Combinatorial Problems*, in R. E. Miller and J. W. Thatcher (editors). Complexity of Computer Computations, 1972, pp. 85–103.

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We want to solve it to use lots of **unavoidable sets** to **reduce** the number of **16-clue sets** we need to consider.

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## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Algorithm: Hitting Sets

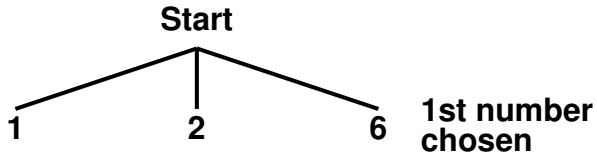
## An example Hitting Set Problem

Let  $M = \{1, 2, 3, \dots, 12\}$  and

$$\mathcal{A} := \left\{ \begin{array}{l} \{1, 2, 6\}, \\ \{2, 3, 4, 8\}, \\ \{1, 7, 8, 9\}, \\ \{3, 4, 6, 9\}, \\ \{4, 6, 10, 12\}, \\ \{2, 10, 11, 12\}, \\ \{5, 7, 8, 9\}, \\ \{5, 7, 10, 12\}, \\ \{1, 3, 4, 5, 11\} \end{array} \right\}$$

Find a 3-subset of  $M$  intersecting all members of  $\mathcal{A}$   
**non-trivially.**

# Backtrack Search in Action



## Sudoku

What is a Sudoku?

## Solving Sudoku

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Backtrack Search in Action

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

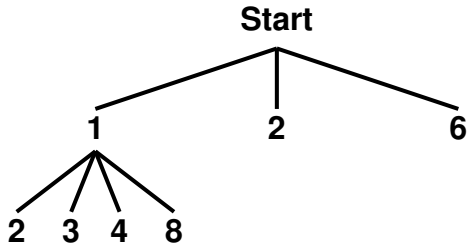
## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search



**1st number  
chosen**

**2nd number  
chosen**

# Backtrack Search in Action

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

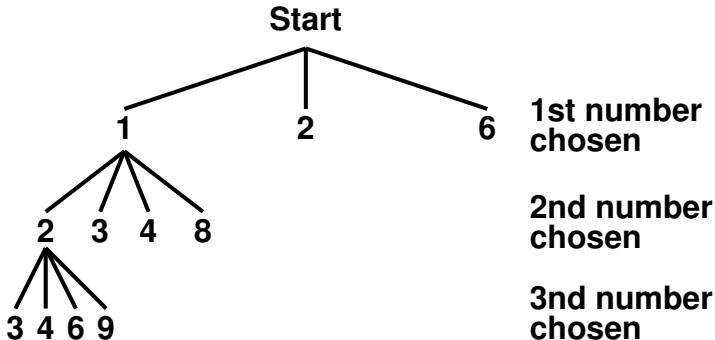
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How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search



# Backtrack Search in Action

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

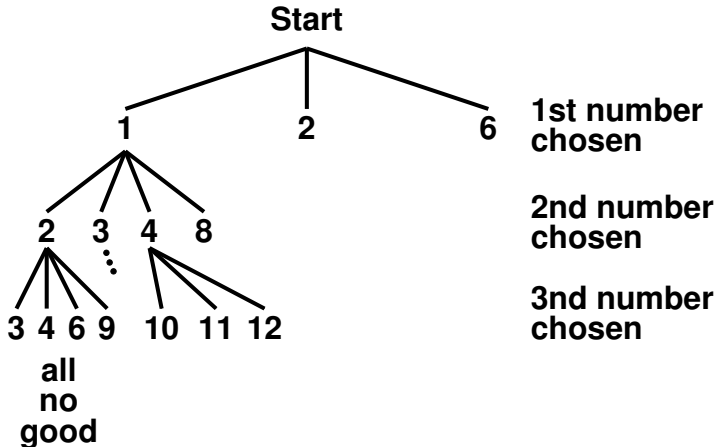
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How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search



## Sudoku

What is a Sudoku?

## Solving Sudoku

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

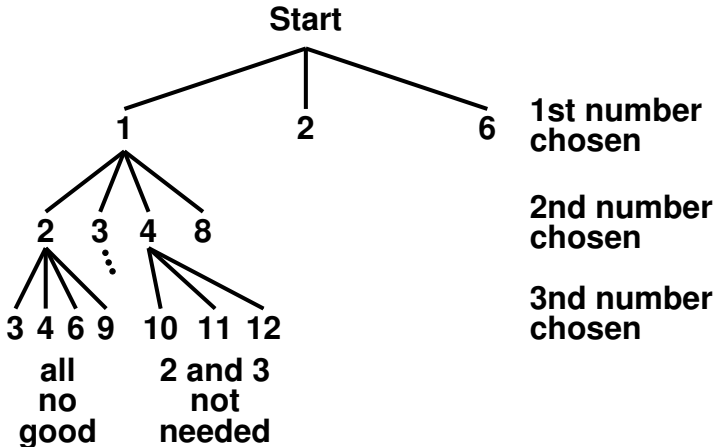
## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search



## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

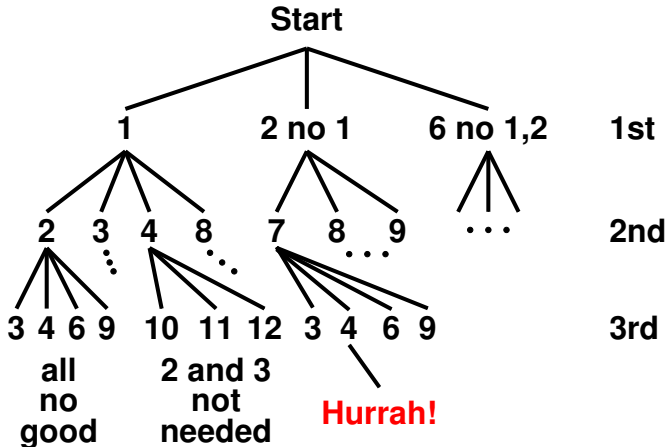
## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search



**Unique solution: {2,4,7}**

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What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

# Performance for our problem

I have run this method on [all 49151](#) solutions of the 17-clue Sudoku puzzles collected by Gordon Royle.

## Sudokus

What is a Sudoku?

## Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

## Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

## Open problems

## 16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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## The good news:

- Usually finds [2048](#) unavoidable sets in  $\approx 10$  s.

Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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Sudokus

What is a Sudoku?

Solving Sudokus

Conclusions

Backtrack search

Sudoku difficulty

Symmetry

How many grids?

Equivalent Sudokus

Symmetry Breaking

Open problems

16 Clues

How many clues?

Unavoidable Sets

The Hitting Set Problem

Backtrack Search

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If I go on like this with the **5 472 730 538**,

**I need another 300 000 CPU years.**