Diagram Monoids and their Congruence Lattices

Nik Ruškuc
nik.ruskuc@st-andrews.ac.uk

School of Mathematics and Statistics, University of St Andrews

Joint work with: James East, James Mitchell, Michael Torpey

The Asia-Australia Algebra Conference
Western Sydney University, 21 January 2019
Prove that every subgroup of the quaternion group $Q_8$ is cyclic. List all subgroups, draw the Hasse diagram, and prove that they are all normal.
Another exercise: $S_n$

Theorem

*The normal subgroups of the symmetric subgroup $S_n$ are 1, $A_n$ and $S_n$.*

Remark

Three exceptions: $S_1$, $S_2$ (too small) and $S_4$ (because of the Klein 4-group $K_4$).
From normal subgroups to congruences

- Normal subgroups $\rightarrow$ quotients $\leftrightarrow$ homomorphic images
- Generally in maths: quotients need equivalence relations
- Congruence: equivalence relation compatible with operations
- E.g. for $N \trianglelefteq G$: $x \sim y \iff xy^{-1} \in N$
- $\text{Cong}(\mathcal{A}) :=$ congruence lattice of $\mathcal{A}$
From permutations to transformations

- \([n] := \{1, \ldots, n\}\)
- \(T_n := \text{all mappings } [n] \rightarrow [n]\)
- Semigroup/monoid under composition of mappings
- Full transformation monoid
- Every semigroup embeds into some full transformation monoid (Cayley Theorem for semigroups)
Theorem (A.I. Malcev 1952)

The full transformation monoid $\mathcal{T}_n$ has $3n - 1$ congruences and they form a chain. ($n \geq 4$)

Remark

$| \text{Cong}(\mathcal{T}_{n+1})| - | \text{Cong}(\mathcal{T}_n)| = 3 = | \text{Cong}(S_{n+1})|$. 
Congruences from ideals

- \( \Delta_X := \{(x, x) : x \in X\} \), \( \nabla_X := X \times X \)
- For \( \alpha \in T_n \) let \( \text{rank} \, \alpha := |\text{im} \, \alpha| \).
- The ideals of \( T_n \) are \( I_r = \{\alpha \in T_n : \text{rank} \, \alpha \leq r\} \).
- All ideals are principal, and they form a chain.
- To every ideal \( I_r \) associate a congruence
  \[ R_r = \nabla_{I_r} \cup \Delta_{T_r} \] (Rees congruence).
- Caution: not every congruence is Rees.
Consider the difference \( I_r \setminus I_{r-1} \):

\[
D_r = \{ \alpha \in T_n : \text{rank} \alpha = r \}
\]

Can be gridded by kernel/image, e.g. for \( n = 4, r = 2 \):

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Each cell (\( \mathcal{H} \)-class) has \( r! \) elements

When image cross-sects kernel, the \( \mathcal{H} \)-class is a subgroup \( \cong S_r \)
For $N \trianglelefteq S_r$ let $R_N$ be the kernel of the following homomorphism:

Also let $\nu_N \coloneqq R_N|_{D_r}$. 
The congruences of the full transformation monoid $T_n$ are precisely:

$$\Delta \subseteq R_1 \subseteq R_{S_2} \subseteq R_2 \subseteq R_{A_3} \subseteq R_{S_3} \subseteq R_3 \subseteq R_{K_4} \subseteq R_{A_4} \subseteq R_{S_4} \subseteq \cdots \subseteq R_{n-1} \subseteq R_{A_n} \subseteq R_{S_n} \subseteq \Delta.$$
Malcev variations

Analogous results for:

- partial transformations (Sutov 1961)
- partial bijections (Liber 1953)
- order preserving, full or partial (Aizenstat 1962)
- ...

\(\text{Cong}(S)\) is a chain in all of them.
From transformations to partitions

Can view mappings graphically:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 2 & 3 & 4 & 4
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1' & 2' & 3' & 4' & 5'
\end{pmatrix}
\]

And then compose:
Partition monoid $\mathcal{P}_n$

Partition = a set partition of $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$

For example: $\alpha = \{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\}$

Composition of partitions:

Warning: This is not the composition of binary relations.
Diagram monoids: submonoids of $\mathcal{P}_n$
From diagram monoids to diagram algebras

Diagram algebras are certain twisted semigroup algebras of the diagram monoids; notable examples:

- Brauer monoid (2-partitions) → Brauer algebra – representation theory of orthogonal groups;
- Temperley–Lieb monoid (planar Brauer) → Temperley–Lieb algebra – statistical mechanics;
- Partition algebra, Motzkin algebra, . . .

Twisted product:

\[ \alpha \cdot \beta = q^{m(\alpha, \beta)} \alpha \beta, \]

where \( q \) is a fixed scalar, and \( m(\alpha, \beta) \) is the number of ‘floating components’ when forming \( \alpha \beta \) in \( \mathcal{P}_n \).

Our question

- Describe the congruences lattice $\text{Cong}(\mathcal{P}_n)$...
- ...as well as the congruence lattices of the important diagram monoids.
The ideal structure of $\mathcal{P}_n$

Useful parameters, defined by example:

$$\alpha = \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
1' & 2' & 3' & 4' & 5' & 6'
\end{array}$$

$\text{dom } \alpha = \{1, 3, 5, 6\}$  
$\text{ker } \alpha = \overline{\alpha} = \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$  
$\text{codom } \alpha = \{1', 4', 6'\}$  
$\text{coker } \alpha = \underline{\alpha} = \{\{1', 6'\}, \{2', 3'\}, \{4'\}, \{5'\}\}$

$\text{rank } \alpha = 2$.

The ideals of $\mathcal{P}_n$ are:

$$I_r = \{\alpha \in \mathcal{P}_n : \text{rank } \alpha \leq r\}.$$

Note: Start at $r = 0$.
Note: Subgroups on each level are again $S_r$.  

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Nik Ruškuc: Congruences of Diagram Monoids
The minimal ideal $I_0$ of $\mathcal{P}_n$

- Rows and columns labelled by equivalence relations on $[n]$
- Subgroups trivial
- A $B_n \times B_n$ rectangular band
- Two incomparable congruences:
  \[
  \rho_0 = \{ (\alpha, \beta) \in I_0 \times I_0 : \overline{\alpha} = \overline{\beta} \}, \\
  \lambda_0 = \{ (\alpha, \beta) \in I_0 \times I_0 : \alpha = \beta \}!!
  \]

- $\rho_0 \cap \lambda_0 = \Delta$, $\rho_0 \lor \lambda_0 = R_0$. 

\[
\begin{array}{c}
\Delta \\
\lambda_0 \\
\rho_0 \\
R_0
\end{array}
\]
The ideal $I_1$ of $\mathcal{P}_n$

A retraction:

$$I_1 \to I_0, \quad \alpha \mapsto \hat{\alpha}, \quad \overline{\alpha} = \overline{\alpha}, \quad \hat{\alpha} = \alpha$$

Leading to congruence: $\mu_1 = \{ (\alpha, \beta) \in I_1 \times I_1 : \hat{\alpha} = \hat{\beta} \} \cup \Delta$.

And some joins:

- $\rho_1 := \rho_0 \vee \mu_1$, $\lambda_1 := \lambda_0 \vee \mu_1$
- $\rho_1 \vee \lambda_1 = R_1$. 
Quotienting $S_2$

- $\rho_{S_2} := \rho_1 \cup \nu_{S_2}$, $\lambda_{S_2} := \lambda_1 \cup \nu_{S_2}$,
  $\mu_{S_2} := \mu_1 \cup \nu_{S_2}$, $R_{S_2} := R_1 \cup \nu_{S_2}$
- After this everything returns to the Malcev pattern (chain).
Theorem

$\text{Cong}(P_n)$ is the lattice shown on the right.

J. East, J.D. Mitchell, NR, M. Torpey,
Congruence lattices of finite diagram monoids,

- In fact, we compute the congruence lattice for all diagram monoids.
- They all have the same basic shape, with differing numbers of diamonds.
Brauer monoid $\mathcal{B}_n$, $n$ even

- $\mathcal{B}_n$ consists of all 2-partitions
- Ideals: $I_r = \{ \alpha : \text{rank } \alpha \leq r \}$, $r$ even
- Retraction $I_2 \rightarrow I_0$:

```
1 2 3 4 5 6
1' 2' 3' 4' 5' 6'
```

- Subgroups on levels 2 and 4: $S_2, S_4$
- Allowable quotients: $S_2, S_2/S_2, S_4, S_4/K_4$
Cong($\mathcal{B}_n$), $n$ even
From finite to infinite: \(\text{Cong}(S_X)\)

**Theorem (Baer 1934 (?))**

The normal subgroups of the symmetric group \(S_X\) (\(X\) infinite) are

- \(S_\kappa = \{\pi : |\text{supp } \pi| < \kappa\}, \ \kappa \in [\aleph_0, |X|]\)
- \(A_\infty\) (alternating group)
- \(\{1\}, S_X\).

Remark: \(\text{Cong}(S_X)\) is a chain.
Cong($\mathcal{T}_X$), $X$ infinite (Malcev 1952)

- $I_{\aleph_0} = \{\alpha : \text{rank } \alpha < \aleph_0\}$ – finitary part
- Below $I_{\aleph_0}$ congruences analogous to $\mathcal{T}_n$
- I.e. Rees congruences and quotienting $S_n$
- Above $I_{\aleph_0}$ another type of congruence:

$$\mu_{\xi}^{\eta} = \{(\alpha, \beta) : \text{rank } \alpha, \text{rank } \beta < \eta, |\alpha \triangle \beta| < \xi\} \cup \Delta$$

- Can take finite unions:

$$R_{\eta} \cup \mu_{\xi_1}^{\eta_1} \cup \mu_{\xi_2}^{\eta_2} \cup \cdots \cup \mu_{\xi_k}^{\eta_k}$$

where $\xi_k < \cdots < \xi_1 \leq \eta < \eta_1 < \cdots < \eta_k = |X|^+$. 
- Only finite unions because $\xi_1, \ldots, \xi_k$ is decreasing and cardinals are well-ordered!
Cong($\mathcal{T}_X$) for $|X| = \aleph_0, \aleph_1, \aleph_2$
Cong(\(P_X\))

- Compared to congruences on \(T_X\), those on \(P_X\) have a further two parameters.
- They ‘measure’ by how much the kernels and cokernels of related elements differ.
- \(\lambda_\zeta := \{(\alpha, \beta) : |\overline{\alpha} \triangle \overline{\beta}| < \zeta\}, \rho_\zeta := \{(\alpha, \beta) : |\alpha \triangle \beta| < \zeta\}\)
- Every point in Cong(\(T_X\)) becomes a grid.
- For instance, congruences contained in \(R_0\) are \(\lambda^1_\zeta := \lambda_\zeta \cap R_0, \rho^1_\zeta := \rho_\zeta \cap R_0\) for \(\zeta \in \{1\} \cup [\aleph_0, |X|^+]\).
- Alternative viewpoint: the square \(\{0, 1\} \times \{0, 1\}\) from the finite case, now becomes \((\{1\} \cup [\aleph_0, |X|^+]) \times (\{1\} \cup [\aleph_0, |X|^+])\).
- These layers gradually taper towards the top of the lattice.
The interval $[\Delta, R_0]$ in $\text{Cong}(\mathcal{P}_X)$
The interval $[\Delta, R_{\aleph_0})$

$$(\eta, N)$$

$$(4, K_4)$$

$$(4, \{\text{id}_4\})$$

$$(3, S_3)$$

$$(3, A_3)$$

$$(3, \{\text{id}_3\})$$

$$(2, S_2)$$

$$(2, \{\text{id}_2\})$$

$$(1, \{\text{id}_1\})$$

Proof of Theorem 6.2. Denote by $\text{Cong}_{1,2}(M_X)$ the subposet of $\text{Cong}(M_X)$ consisting of all congruences of type (CT1) or (CT2), and by $\text{Cong}_3(M_X)$ that of all congruences of type (CT3). We will prove that each of these two sets is wqo, and the theorem will then follow from Lemma 6.3.

We begin with $\text{Cong}_{1,2}(M_X)$. By Theorem 6.1 (i) it is isomorphic to $C_{1,2} = \{(n, N, \downarrow_1, \downarrow_2) : n = 1, 2, N \in S_n, \downarrow_1, \downarrow_2 \in \{\text{id}\}\} \subseteq \{\eta, N\}$. The subposet is then ordered by $(n, N, \downarrow_1, \downarrow_2) \succeq C_{1,2}(n_0, N_0, \downarrow_{0,1}, \downarrow_{0,2})$, $n < n_0$ or $(n = n_0$ and $N \succeq N_0))$, and $\downarrow_1 \succeq \downarrow_{0,1}, \downarrow_2 \succeq \downarrow_{0,2}$.
Whole of $\text{Cong}(\mathcal{P}_X)$ when $|X| = \aleph_2$
Properties of Cong($\mathcal{P}_X$)

- Cong($\mathcal{P}_X$) is distributive.
- Cong($\mathcal{P}_X$) is well-quasi-ordered (no infinite descending chains or infinite antichains).
- Every congruence can be generated by at most $|X|$ pairs.
- Complete characterisation of minimal generating sets.
- Cong($\mathcal{P}_X$) $\cong$ Cong($\mathcal{PB}_X$).

Current work

- January 2019: a unified theoretical framework for computing the congruence lattices in the finite case.
- Starts from a small ideal \( r \in \{0, 1, 2, 3, 4\} \) and builds up inductively via a series of ideal extensions.
- Complete description of \( \text{Cong}(I) \) where \( I \) is an arbitrary ideal of \( T_X, P_X, B_X \), etc.
Future work

- Compute congruence lattices of one-sided ideals.
- Investigate the congruence lattice of the twisted partition monoid $P_T^n = \mathbb{N}_0 \times P_n$:
  
  $$(k, \alpha)(l, \beta) = (k + l + m(\alpha, \beta), \alpha \beta),$$

  where $m(\alpha, \beta)$ is the number of connected components that ‘get lost in the middle’ when forming the product $\alpha \beta$ (suggested by W. Mazourchuk).
- Can these results be applied to say something useful/interesting (e.g. ideals, representations, etc.) about diagram algebras?

Partially funded by: EPSRC EP/S020616/1, Diagram Monoids and Their Congruences
THANK YOU!

- To James East and all my collaborators: for sharing the journey and all the work.
- To James East and Mathematics Department at WSU: for your hospitality.
- To you: for listening.