Classes of permutations avoiding 231 or 321

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Aim

- Introduce the area of pattern classes of permutations...
- ...by relating a single, fairly specific topic/story...
- ...while keeping an eye on general motifs and links with other areas.
- Emphasise:
  - Importance of structure and links with language theory in approaching ‘classical’ combinatorial problems, e.g. enumeration or nature of generating functions.
  - Importance of the concept of partial well-order.
Sorting by a stack
Sorting by a stack

31524
Sorting by a stack

[Diagram of sorting by a stack with numbers 3, 1524]
Sorting by a stack
Sorting by a stack
Sorting by a stack

1

3

5

24

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1

2

3

5

2

3

4
Sorting by a stack

12

3

5

4
Sorting by a stack

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Sorting by a stack
Proposition

A permutation $\sigma$ can be sorted by a stack if and only if $\sigma$ does not contain a subsequence $\ldots a \ldots b \ldots c \ldots$ with $c < a < b$. 
Permutation poset $S$

- $S = \{1, 12, 21, 123, 132, 213, 231, 312, 321, 1234, \ldots \} \rightarrow$ all finite permutations.

- Pattern involvement ordering:

  \[ \sigma \leq \tau \iff \tau \text{ contains a subsequence order-isomorphic to } \sigma \]

- E.g. $231 \leq 31524$, $321 \nleq 31524$. 
Pattern classes and avoidance

- Downward closed set $C$: $\sigma \in C$ & $\tau \leq \sigma \Rightarrow \tau \in C$.
- $C$ is a down-set iff

  $$C = \text{Av}(B) = \{\sigma \in S : (\forall \beta \in B)(\beta \not\prec \sigma)\}$$

  for some (unique antichain) $B$.
- Call $B$ the basis of $C$.
- $C$ is finitely based if $B$ is finite.
Geometric and relational structures viewpoints
A permutation (31524) can be viewed as a set of points in the plane
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Geometric and relational structures viewpoints

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![Diagram of points in a plane]
A permutation (31524) can be viewed as a set of points in the plane

in which case involvement is just taking subsets.
A permutation (31524) can be viewed as a set of points in the plane:

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
\quad a & c & e & d & b
\end{array}
\]

in which case involvement is just taking subsets.

Or it can be viewed as a relational structure with two linear orders:

\[
\left(\{a, b, c, d, e\}, \ a < b < c < d < e, \ b \prec d \prec a \prec e \prec c\right)
\]
A permutation (31524) can be viewed as a set of points in the plane

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\begin{array}{c}
  a \\
  b \\
  c \\
  d \\
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\]

and involvement is embeddability of structures.
What is asked about a pattern class?

- Enumeration sequence: \( c_n = |\{\sigma \in C \mid |\sigma| = n\}| = ? \)

- Generating function: \( f(x) = \sum_{n=1}^{\infty} c_n x^n \).
  Is it perhaps: (a) rational? (b) algebraic?
  (c) D-finite? (d) worse?

- Is \( C \) finitely based?

- Structure of \( C \) and its permutations?
Flavour of the field: some sample results

Theorem (Bona 1997)

*The generating function for $C = \text{Av}(1342)$ is*

$$\frac{32x}{-x^2 + 20x + 1 - (1 - 8x)^{3/2}}.$$ 

Theorem (Regev 1981; Gessel 1990)

*The growth rate of $\text{Av}(12\ldots r)$ is $(r - 1)^n$.*

Theorem (Simion, Schmidt 85; West 96)

*Complete enumeration of $\text{Av}(\alpha, \beta)$, $|\alpha| = 3$, $|\beta| = 3, 4$.*

Theorem (Albert, Atkinson 2005)

*If a class $C$ contains only finitely many simple permutations then $C$ is partially well ordered and its generating function is algebraic.*
Classes $\text{Av}(\beta), \ |\beta| = 3$

Fact

The symmetry group of $S$ is isomorphic to the dihedral group $D_8$.

Fact

There are precisely two orbits of permutations of length 3; 231 and 321 are their representatives.
Av(231) – first look

D.E. Knuth, The Art of Computer Programming
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\[ \sigma \quad \tau \]
Av(231) – first look

D.E. Knuth, The Art of Computer Programming

\[ \sigma \in \text{Av}(231) \text{ if and only if } \sigma \text{ can be sorted by a stack.} \]
Av(231) – first look

D.E. Knuth, The Art of Computer Programming

$\sigma \in \text{Av}(231)$ if and only if $\sigma$ can be sorted by a stack.

There are precisely $C_n$ (the $n$th Catalan number) permutations of length $n$ in Av(231).
Av(231) – first look

D.E. Knuth, The Art of Computer Programming

- $\sigma \in \text{Av}(231)$ if and only if $\sigma$ can be sorted by a stack.
- There are precisely $C_n$ (the $n$th Catalan number) permutations of length $n$ in $\text{Av}(231)$.
- Algebraic generating function: $\frac{1 - \sqrt{1 - 4x}}{2x}$. 
Av(321) – first look
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- $\sigma \in \text{Av}(321)$ if and only if $\sigma$ consists of two increasing sequences.
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Av(321) – first look

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- Enumeration: Catalan numbers.
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Av(321) – first look

- $\sigma \in \text{Av}(321)$ if and only if $\sigma$ consists of two increasing sequences.
- Enumeration: Catalan numbers.
- $\sigma \in \text{Av}(321)$ if and only if $\sigma$ can be drawn on two parallel lines.
Partial well order (PWO)

Definition
A partially ordered set \((P, \leq)\) is PWO if it has (no infinite descending chains and) no infinite antichains.

Proposition
The following are equivalent for a countable poset \((P, \leq)\) (with no infinite descending chains):

(i) \(P\) is PWO.

(ii) Every down-set of \(P\) is finitely based (defined by avoidance of finitely many elements).

(iii) \(P\) has only countably many downsets.
Higman’s Lemma

G. Higman, Ordering by divisibility in abstract algebras, Proc. LMS 3 (1952)

Theorem (Short version)

The free monoid $X^*$ of finite rank is PWO with respect to the subword (=subsequence) ordering.

Theorem (Long version, abridged)

Let $\mathcal{A} = (A, F, \leq)$ be an ordered algebraic structure and let $X$ be a generating set. In the presence of some natural compatibility conditions between $F$ and $\leq$, we have

$\mathcal{A}$ is PWO $\iff$ $X$ is PWO.
Corollary

Every down-set of $X^*$ is regular.
(in the language-theoretic sense: FSA, regular expressions)

Corollary

The generating function of a down-set in $X^*$ is rational.

For almost all families of combinatorial objects with a rational GF, it is easy to foresee that there will be a bijection between these objects and words of a regular language. (Bousquet-Mélou, 2006)
An easy application: concatenation of two increases

\[ C = \text{Av}(321, 3142, 2143). \]

- Encoding into \( \{A, B\}^* \) – order preserving; hence: PWO.
- Encoding with uniqueness: \( \{A, B\}^* \setminus A^*B^+ \).
- Hence: rational generating function for \( C \) and all its subclasses.

Av(231) & Av(321): PWO or not PWO?
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\[(\sigma, \tau) \mapsto \tau\]
Av(231) & Av(321): PWO or not PWO?

Proposition

Av(231) is PWO.
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Proposition

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Proposition

Av(321) is not PWO.
Some corollaries
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Proposition (Folklore)

\( \text{Av}(231) \) has only countably many subclasses and they are all finitely based.
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Theorem (Albert, Atkinson 2005)

Every proper subclass of Av(231) has a rational generating function.
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Proposition (Folklore)

Av(321) has uncountably many subclasses with uncountably many different generating functions.
(Finite, geometric) grid classes

GGC: finite grid, with a diagonal line in each cell (or empty).


Theorem
Every subclass of a geometric grid class is finitely based, PWO and has a rational generating function.
Relative position of two points in non-adjacent cells is completely determined by their cells (SW-NE).

Points in adjacent cells behave as in $\text{Av}(321, 3142, 2143)$ (up to symmetry).
Key observation
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If a subclass $C \subseteq \text{Av}(321)$ contains these configurations of arbitrary ‘width’ and ‘length’ then in fact $C = \text{Av}(321)$. 
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- If a subclass $C \subseteq \text{Av}(321)$ contains these configurations of arbitrary ‘width’ and ‘length’ then in fact $C = \text{Av}(321)$.
- Otherwise, there exists $n \in \mathbb{N}$ such that elements of $C$ can be encoded by successively encoding $n$ consecutive cells, and only finite amount of additional information.

**Theorem**

*Every finitely based proper subclass of $\text{Av}(321)$ has a rational generating function.*
Subclasses of Av(321)


Theorem
Every finitely based proper subclass of Av(321) has a rational generating function.

Theorem
Every PWO subclass of Av(321) has a rational generating function.
Can a general theory of infinite geometric grid classes of permutations be developed?
Can a general theory of infinite geometric grid classes of permutations be developed?
Vistas

- Can a general theory of infinite geometric grid classes of permutations be developed?

- What ‘sparseness’ and ‘regularity’ conditions should be imposed?
Open Problem
Is it decidable whether a finitely based permutation class $\text{Av}(\beta_1, \ldots, \beta_k)$ is PWO?


Open Problem

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THANK YOU!