Permutations and words

Nik Ruškuc
nik@mcs.st-and.ac.uk

School of Mathematics and Statistics, University of St Andrews

Linz, 18 November 2013
Involvement of permutations

\[231 \leq 631524\]
\[231 \not\leq 54132\]

An infinite poset \( \mathcal{S} \).
Well founded: no infinite descending chains.
Involvement of permutations

\[ 231 \leq 631524 \]
\[ 231 \not\leq 54132 \]

An infinite poset \( S \).
Well founded: no infinite descending chains.
Involvement of permutations

\[231 \leq 631524\]
\[231 \not\geq 54132\]

An infinite poset \(S\).

Well founded: no infinite descending chains.
Involvement of permutations

\[ 231 \leq 631524 \]
\[ 231 \not\leq 54132 \]

An infinite poset \( S \).

Well founded: no infinite descending chains.
Involvement of permutations

\[ 231 \leq 631524 \]
\[ 231 \not\leq 54132 \]

An infinite poset \( S \).

Well founded: no infinite descending chains.
Involvement of permutations

\[ 231 \leq 631524 \]
\[ 231 \not\leq 54132 \]

An infinite poset \( S \).

Well founded: no infinite descending chains.
Involvement of permutations

\[ 231 \leq 631524 \]
\[ 231 \not\geq 54132 \]

An infinite poset \( S \).

Well founded: no infinite descending chains.
Poset $S$

12345
1234 1243 1324 1342
123 132 213 231 312 321
12345 1243 1324 1342
12 21
1
Pattern class of permutations $\mathcal{C}$: a downward closed set (ideal) under $\leq$; i.e. $\sigma \leq \tau \in \mathcal{C} \Rightarrow \sigma \in \mathcal{C}$.

Basis $B = B(\mathcal{C})$: the minimal permutations not in $\mathcal{C}$.

$\mathcal{C} = \{\sigma : (\forall \beta \in B)(\beta \not\leq \sigma)\} = \text{Av}(B)$.

Bases = antichains in $S$.

Not partially well ordered = infinite antichains = infinitely based pattern classes.

Context: sorting mechanisms (stacks, queues); mathematical biology; combinatorics of relational structures.
Theory of Pattern Classes

What can be asked of a pattern class $C$?

- Enumeration: $c_n = |C_n|$ = the number of permutations of length $n$ in $C$.
- Generating function $\sum_{n=1}^{\infty} c_n x^n$: rational, algebraic, holonomic?
- Asymptotic behaviour: estimate $c_n$ as $n \to \infty$.
- Basis: finite, infinite, size?
- Order theoretic properties: antichains, pwo, join property, . . .
- Algorithmic problems: membership, complexity, computability of the basis, . . .
- Model theoretic properties: simple permutations, decompositions, structure, . . .
Sample results

**Theorem (folklore)**

For any \( C = \text{Av}(\pi) \), \( |\pi| = 3 \), we have

\[
|C_n| = \frac{1}{n+1} \binom{2n}{n}.
\]

**Theorem (Regev 1981; Gessel 1990)**

If \( C = \text{Av}(12\ldots r) \) then

\[
|C_n| \sim (r-1)^n.
\]

**Theorem (Bona 1997)**

The generating function for \( C = \text{Av}(1342) \) is

\[
\frac{32x}{-x^2 + 20x + 1 - (1 - 8x)^{3/2}}.
\]
Wilf–Stanley Conjecture (= Markus–Tardos Theorem)

Theorem (Markus, Tardos 2004)

For every pattern class $C \neq S$ there exists $q \in \mathbb{R}$ such that

$|C_n| \leq q^n.$

Conjecture

$n \sqrt{|C_n|}$ has a limit as $n \to \infty.$
Do permutations, subject to pattern avoidance restrictions, behave like words in some sense?
Words

$A$ – an alphabet; $A^*$ – words over $A$.

Subword ordering: $abac \leq cacbabca$, $abac \not\leq bacbca$.

[Word = sequence; subword = subsequence.]

Downward closed set: $u \leq v \in X \Rightarrow u \in X$. 
Words

$A$ – an alphabet; $A^*$ – words over $A$.

Subword ordering: $abac \leq cacbabc$, $abac \not\leq bacbca$.

[Word = sequence; subword = subsequence.]

Downward closed set: $u \leq v \in X \Rightarrow u \in X$. 
Words

$A$ – an alphabet; $A^*$ – words over $A$.

Subword ordering: $abac \leq cacbabca$, $abac \nless bacbca$.

[Word = sequence; subword = subsequence.]

Downward closed set: $u \leq v \in X \Rightarrow u \in X$. 
Poset $\mathcal{W}$
Higman’s Theorem

Theorem (Higman 1952)
$A^*$ is PWO under $\leq$ (no infinite antichains).

Corollary
Every downward closed set has a finite basis.

Corollary
Every downward closed set $W$ can be expressed as

$$W = A^* \setminus \bigcup_{i=1}^{n} A^* a_{i,1} A^* a_{i,2} A^* \ldots A^* a_{i,l_i} A^*.$$

Corollary
Every downward closed set is regular.
Regular languages

Theorem (Kleene)

A language is regular iff it is accepted by a finite state automaton.

\[
p_n = q_{n-1} + r_{n-1} \\
q_n = p_{n-1} + s_{n-1} \\
r_n = p_{n-1} + s_{n-1} \\
s_n = r_{n-1} + q_{n-1}
\]

Corollary

Every regular set has a rational generating function, which can be effectively computed.
Back to permutations

Can we encode permutations by words, while preserving the nice properties of the subword ordering?
Rank encoding


Replace every entry by the number of smaller entries after it +1.
Example: $\pi = 2451637$, $\rho(\pi) = 2331211$.

In general: need an infinite alphabet. But for some classes finite alphabet suffices.

$\Omega_k = \{ \sigma \in S : \rho(\sigma) \in [k]^* \}$ – a pattern class!

Example

$\mathcal{C} = \text{Av}(\{321, 312\})$. For every entry there is at most one smaller entry to the right of it; $\mathcal{C} = \Omega_2$.

Example

Permutations generated by a system with finite memory (a graph).
Rank encoding


Replace every entry by the number of smaller entries after it +1.

Example: $\pi = 2451637$, $\rho(\pi) = 2331211$.

In general: need an infinite alphabet. But for some classes finite alphabet suffices.

$\Omega_k = \{\sigma \in S : \rho(\sigma) \in [k]^*\}$ - a pattern class!

Example

$C = \text{Av} \{321, 312\}$. For every entry there is at most one smaller entry to the right of it; $C = \Omega_2$.

Example

Permutations generated by a system with finite memory (a graph).
Rank encoding


Replace every entry by the number of smaller entries after it +1.

Example: $\pi = 2451637$, $\rho(\pi) = 2331211$.

In general: need an infinite alphabet. But for some classes finite alphabet suffices.

$\Omega_k = \{\sigma \in S : \rho(\sigma) \in [k]^*\}$ – a pattern class!

Example

$C = \text{Av}(\{321, 312\})$. For every entry there is at most one smaller entry to the right of it; $C = \Omega_2$.

Example

Permutations generated by a system with finite memory (a graph).
Rank encoding


Replace every entry by the number of smaller entries after it +1.

Example: $\pi = 2451637$, $\rho(\pi) = 2331211$.

In general: need an infinite alphabet. But for some classes finite alphabet suffices.

$\Omega_k = \{ \sigma \in S : \rho(\sigma) \in [k]^* \}$ – a pattern class!

Example

$C = \text{Av}(\{321, 312\})$. For every entry there is at most one smaller entry to the right of it; $C = \Omega_2$.

Example

Permutations generated by a system with finite memory (a graph).
Rank encoding


Replace every entry by the number of smaller entries after it +1.

Example: \( \pi = 2451637 \), \( \rho(\pi) = 2331211 \).

In general: need an infinite alphabet. But for some classes finite alphabet suffices.

\[ \Omega_k = \{ \sigma \in S : \rho(\sigma) \in [k]^* \} \] – a pattern class!

Example

\( \mathcal{C} = \text{Av} (\{321, 312\}) \). For every entry there is at most one smaller entry to the right of it; \( \mathcal{C} = \Omega_2 \).

Example

Permutations generated by a system with finite memory (a graph).
Involvement and rank encoding

Difficulty: Subpermutations don’t correspond to subwords.
\[ \rho(2451637) = 2331211, \rho(234156) = 222111. \]

Hence: Pattern classes not encoded by downward closed sets of words.

Fortunately, limited damage:

**Proposition**

The set
\[ \{ (\rho(\sigma), \rho(\tau)) : \sigma, \tau \in \Omega_k, \sigma \leq \tau \} \]

is recognised by a transducer (finite state translator).
Rank encoding: results

Theorem
Let $C \subseteq \Omega_k$, and let $B$ be its basis. Then $\rho(C)$ is regular iff $\rho(B)$ is regular. In particular all finitely based subclasses of $\Omega_k$ are regular.

Corollary
If $C$ is a finitely based (or indeed regular) subclass of $\Omega_k$ then:

(i) the generating function of $C$ is rational;
(ii) it can be effectively computed from the basis of $C$;
(iii) the membership problem in $C$ is decidable in linear time.

Follow-on development: Insertion encoding; Albert, Linton, NR, 2005.
Gridding a permutation

3, 5, 4, 6, 9, 2, 11, 12, 1, 10, 8, 7
Gridding a permutation

3, 5, 4, 6, 9, 2, 11, 12, 1, 10, 8, 7
Gridding a permutation

3, 5, 4, 6, 9, 2, 11, 12, 1, 10, 8, 7

\[ M = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \]
Geometric grid classes


\[
\begin{array}{|cc|}
\hline
1 & -1 \\
\hline
1 & -1 \\
\hline
\end{array}
\]
Geometric grid classes

Geometric grid classes

Geometric grid classes

Geometric grid classes


1352764
Geometric grid classes: examples

Example
(1 1) defines the class of juxtapositions of two increasing permutations.

Example (Atkinson 1999)
Av(321, 2143) is the union of geometric grid classes of (1 1) and (1 1)$^T$.

Example (Murphy 2003)
Av(132, 4312) is the geometric grid class of

$$
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix}.
$$

Av(132, 4312) = 

Av(132, 4312) = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix}.
Natural encoding
Natural encoding: properties

- maps words to permutations;
- order preserving (subwords $\mapsto$ subpermutations);
- onto (every word has a code);
- finite to one;
- obstacles to 1 − 1: non-uniqueness of griddings; independent cells;
- can be resolved without leaving regular languages.
Geometric grid classes: results

Theorem
The following hold for every geometric grid class $C$:

(i) $C$ can be encoded by a regular language in a bijective and order-preserving manner;

(ii) $C$ is finitely based;

(iii) $C$ is partially well ordered;

(iv) $C$ has a rational generating function.

Theorem
Every subclass of a geometric grid class is a finite union of geometric grid classes and has all the above properties.
Extension: inflations

\[ 3142[12,21,132,1] = 45216873 \]

\[ C[D] = \{ \sigma[\delta_1, \ldots, \delta_m] : \sigma \in C, \ |\sigma| = m, \ \delta_i \in D \}. \]

\( C \) strongly rational: all subclasses have rational general functions (Albert, Atkinson, Vatter, 2012).

**Theorem (Albert, NR, Vatter)**

*If \( C \) is a geometric grid class, and \( D \) is strongly rational, then \( C[D] \) is strongly rational.*
Application: small classes

Theorem (Vatter 2011)

The unique real root $\kappa \approx 2.20557$ of $x^3 - 2x^2 - 1$ is the largest real number such that there are only countably many pattern classes with growth rate $< \kappa$ (small classes).

For each small class $C$ there exists a geometric grid class $G$ and $k > 0$ such that

$$C \subseteq G[G[\ldots G[G]]\ldots]^{k}$$

Theorem (Albert, NR, Vatter)

Every small pattern class has a rational generating function.
Problems

Conjecture
Are the properties of geometric grid classes (e.g. the basis, enumeration function, etc.) algorithmically computable from the gridding matrix?

Conjecture
The property of being geometrically griddable is algorithmically decidable from basis.

Conjecture
Every grid class has a finite basis and an algebraic generating function.
Thank you

