# Monoids and Their Cayley Graphs

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# Instead of an Apology

*Then, as for the geometrical analysis of the ancients and the algebra of the moderns, besides the fact that they extend to very abstract matters which seem to be of no practical use, the former is always so tied to the inspection of figures that it cannot exercise the understanding without greatly tiring the imagination, while, in the latter, one is so subjected to certain rules and numbers that it has become a confused and obscure art which oppresses the mind instead of being a science which cultivates it.*

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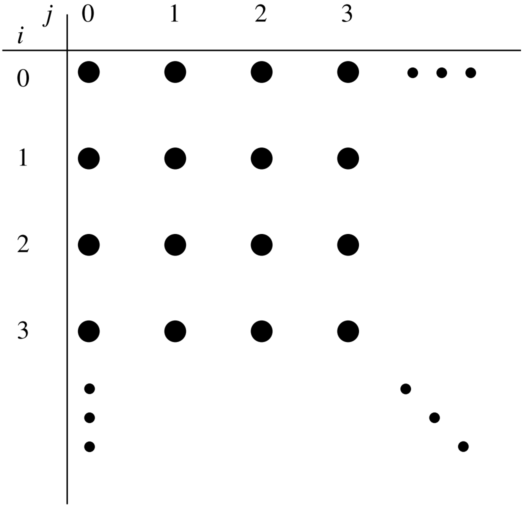

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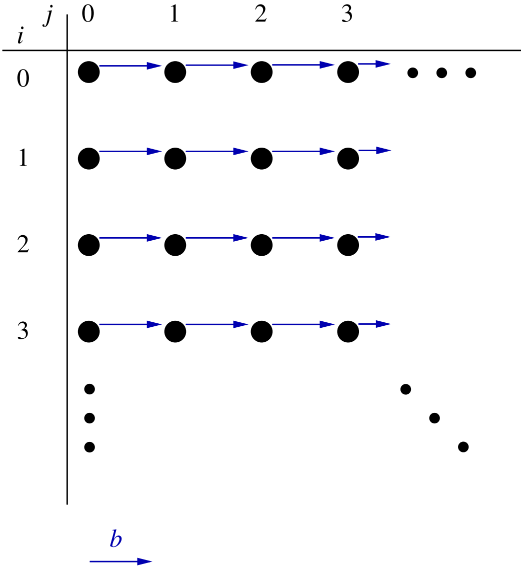

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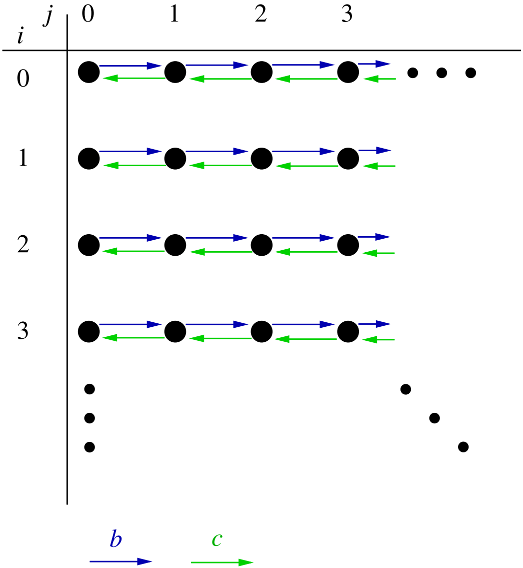

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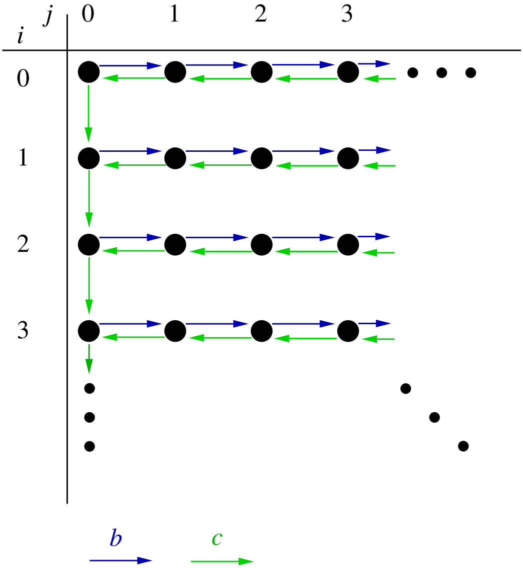

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One more relation: $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

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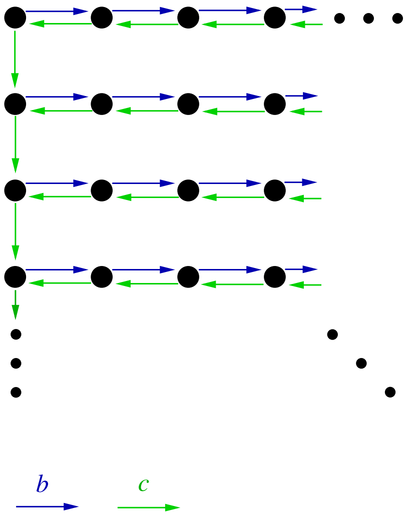

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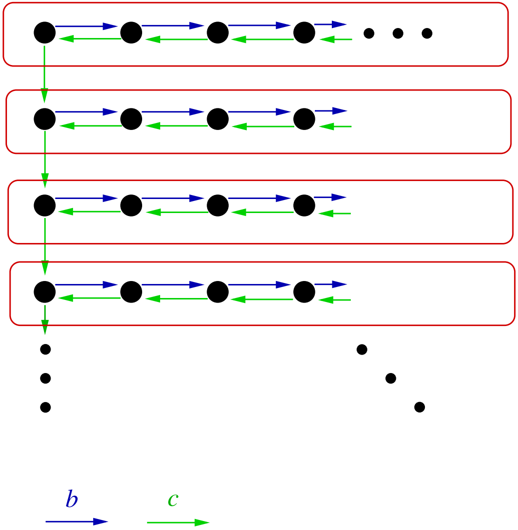

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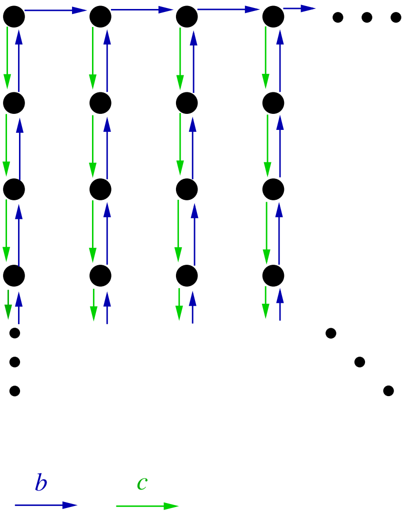

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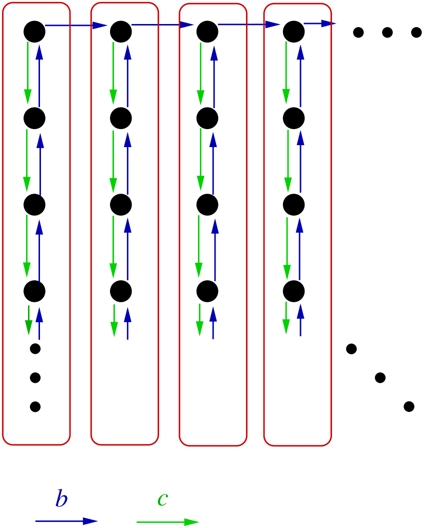

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There is an analogous construction for the left Schützenberger group $\Sigma_l(H)$ of $H$.

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A monoid is a 'union' of groups, many of which are isomorphic, and some of which are actually in the monoid.

# Example

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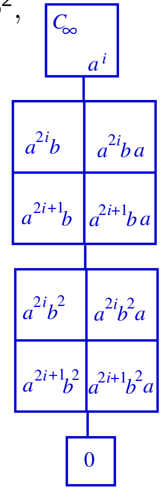

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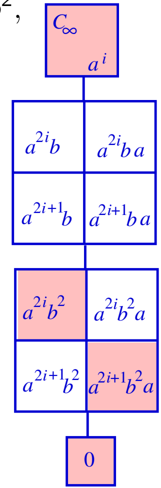

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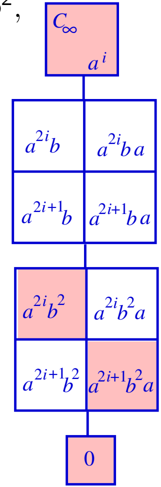

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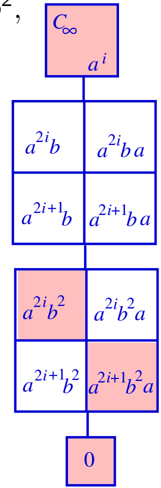

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- Main difference between groups and monoids: poset aspect, direction, (ir)reversibility.
- Many concepts/viewpoints of combinatorial/geometrical/computational group theory have natural extensions in the 'world' of monoids.

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  - ...
- Even if considering Cayley graphs and related topics in this more general setting of monoids may not yield solutions of the hard group-theoretic problems, it may aid understanding of the fundamental concepts involved.

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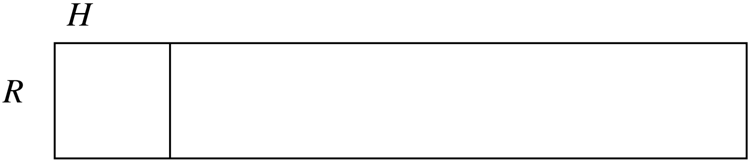

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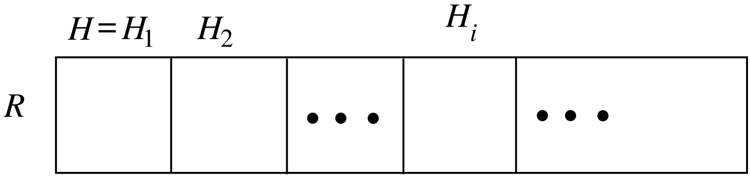


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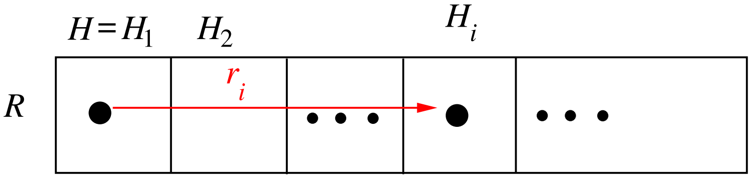


$H_i$ $(i \in I)$ – the $\mathcal{H}$-classes in $R$ $(H_1 = H)$.
$r_i \in M$ $(i \in I)$ – 'representatives': $Hr_i = H_i$.

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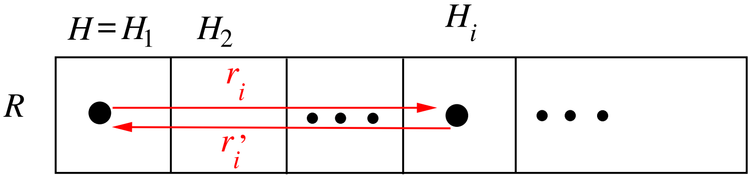


$H_i$ $(i \in I)$ – the $\mathcal{H}$-classes in $R$ $(H_1 = H)$.
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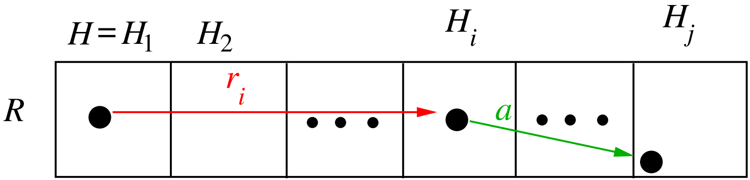


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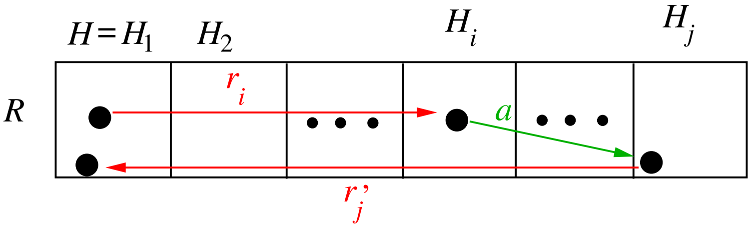


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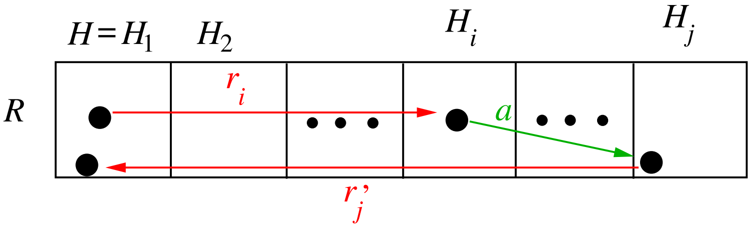


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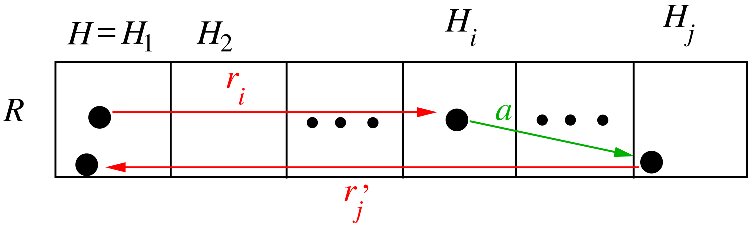


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### Remark

Observe a close analogy with the Schreier generating set for a subgroup of a group.

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**Corollary**

*Let $M$ be a monoid with finitely many $\mathcal{H}$-classes (i.e. finitely many left and right ideals). Then $M$ is finitely generated if and only if all its Schützenberger groups are finitely generated.*

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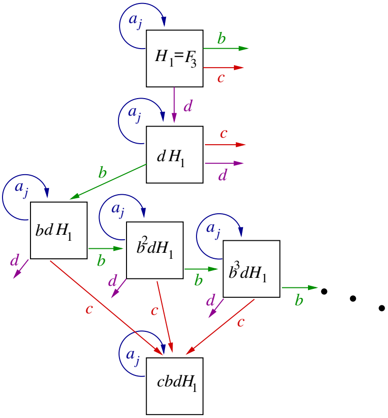

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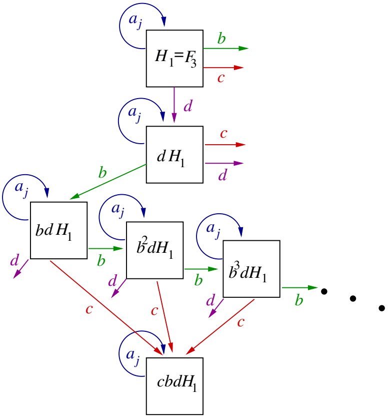


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*$H$ has index 1;*
*its Schützenberger group is*

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i}\ (i = 0, 1, 2, \ldots) \rangle.$$

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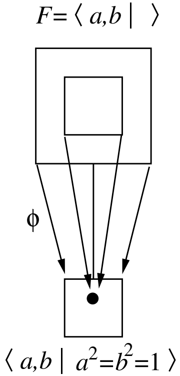


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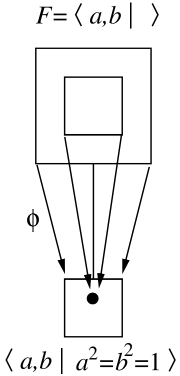


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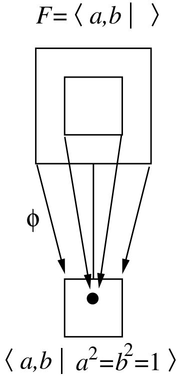


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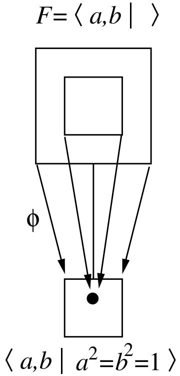


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Suppose $M$ is a monoid with finitely many left and right ideals. If $M$ is automatic, are all its Schützenberger groups automatic? If $M$ is hyperbolic, are all its Schützenberger groups hyperbolic?

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**Theorem**

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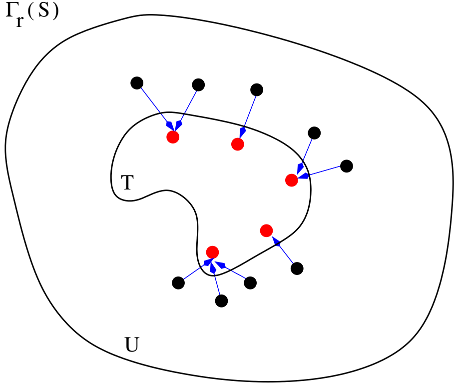

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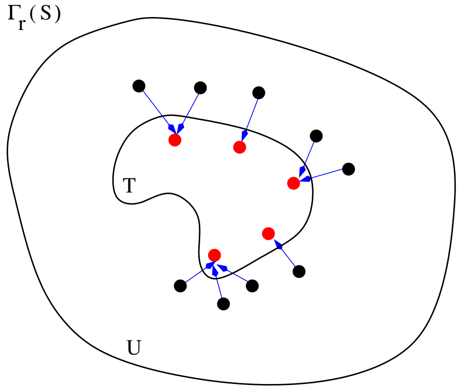


Boundary := left boundary ∪ right boundary.

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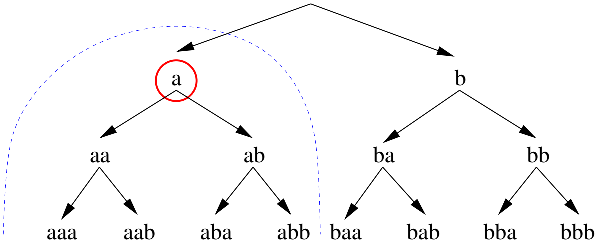

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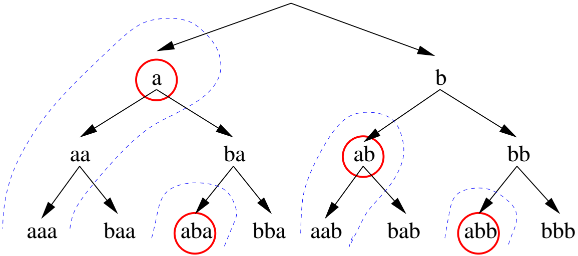

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### Proposition

*Having a finite boundary is independent of the choice of (finite) generating set.*

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**Questions**

How about other properties: FDT, automaticity, etc.?

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*Let $M = \bigcup_{i \in I} S_i$ be a finite disjoint union of subsemigroups. If each $S_i$ is finitely presented and has a finite boundary in $M$ then $M$ is finitely presented as well.*

# A Strange Union

## Proposition (Araujo, NR)

*If $S$ is the four element semigroup*

| $S$ | $a$ | $b$ | $c$ | $0$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $c$ | $0$ |
| $b$ | $b$ | $b$ | $c$ | $0$ |
| $c$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |

*then $S \times C_\infty$ is not finitely presented, but is a disjoint union of two finitely presented subsemigroups $\{a\} \times C_\infty (\cong C_\infty)$ and $\{b, c, 0\} \times C_\infty$.*

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A promising candidate for $\mathcal{P}$: residual finiteness.