Monoids and Their Cayley Graphs

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Instead of an Apology

Then, as for the geometrical analysis of the ancients and the algebra of the moderns, besides the fact that they extend to very abstract matters which seem to be of no practical use, the former is always so tied to the inspection of figures that it cannot exercise the understanding without greatly tiring the imagination, while, in the latter, one is so subjected to certain rules and numbers that it has become a confused and obscure art which oppresses the mind instead of being a science which cultivates it. (Descartes, 1637)

Contents

Cayley Graphs

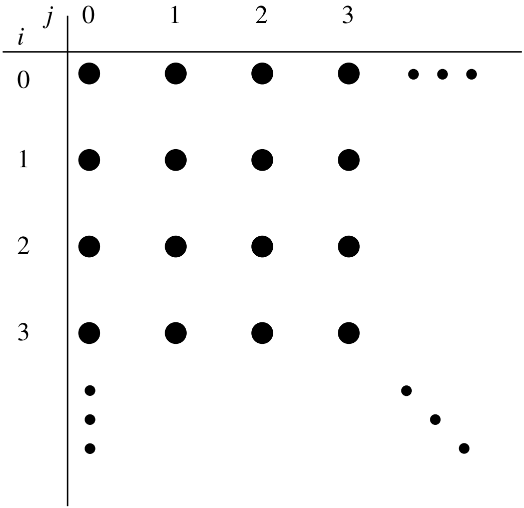
$M = \langle A \rangle$ – a monoid, with a generating set.
The Cayley graph of M w.r.t. A:

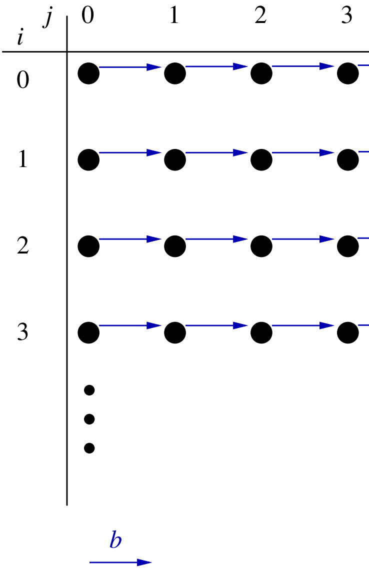
- vertices: M;
- edges: $x \xrightarrow{a} xa$, $x \in M$, $a \in A$.

Notation: $\Gamma_r(M, A) = \Gamma_r(M) = \Gamma(M)$.
'r' stands for 'right'.

Example: The Bicyclic Monoid B

$$B = \langle b, c \mid bc = 1 \rangle = \{c^i b^j \;:\; i, j \geq 0\}.$$





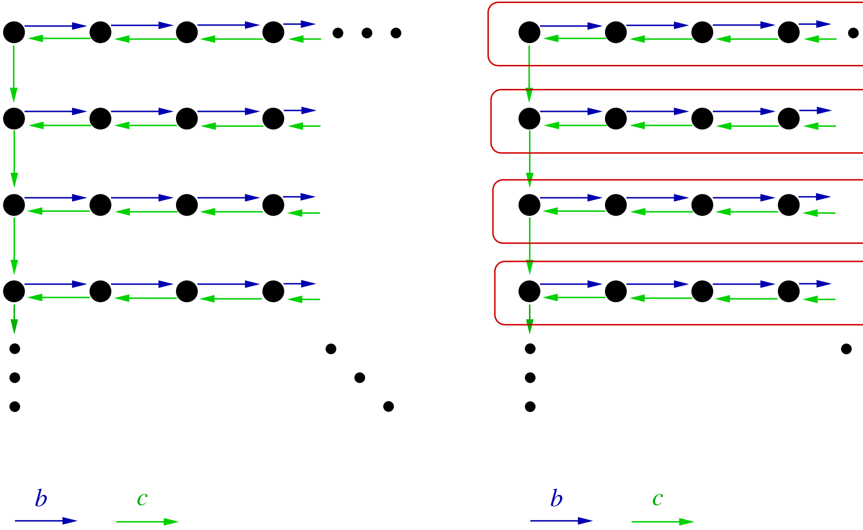
Green's $\mathcal{R}$-equivalence

$x \geq_{\mathcal{R}} y$ iff there is a directed path from x to y in $\Gamma(M)$ iff $xM \supseteq yM$.
$\geq_{\mathcal{R}}$ is a pre-order.
$x\mathcal{R}y \Leftrightarrow x \geq_{\mathcal{R}} y$ & $y \geq_{\mathcal{R}} x \Leftrightarrow xM = yM$.
R_x - the $\mathcal{R}$-class of x ($=$ the strongly connected component of x).
$R_x \geq R_y \Leftrightarrow x \geq_{\mathcal{R}} y$ - partial order.
A monoid is a partially ordered set of $\mathcal{R}$-classes.

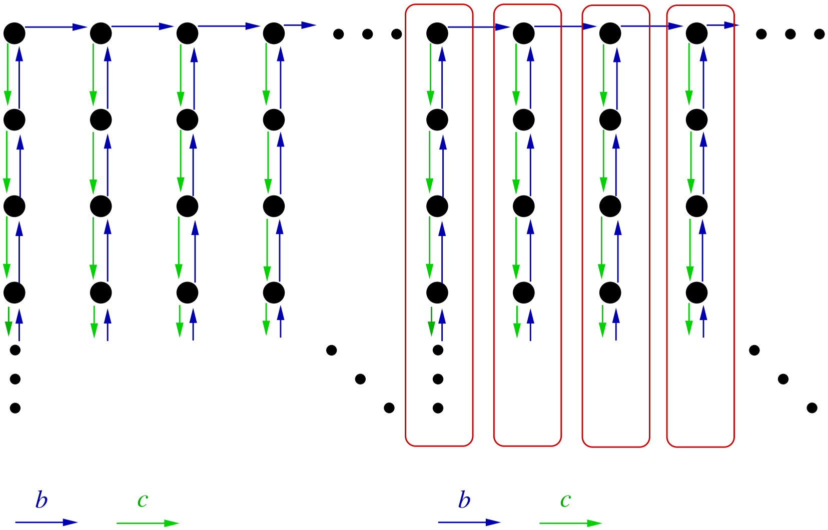
Left/Right Duality

Left Cayley graph $\Gamma_l(M, A)$: $x \xrightarrow{a} ax$, $x \in M$, $a \in A$.
Strongly connected components: $\mathcal{L}$-classes.
$x\mathcal{L}y \Leftrightarrow Mx = My$.
A monoid is a partially ordered set of $\mathcal{L}$-classes.
One more relation: $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

$\mathcal{R}$-Classes of B



$\mathcal{L}$-Classes of B



Schützenberger Group

M – a monoid; H – an $\mathcal{H}$-class.
Stabiliser: $\mathrm{Stab}_r(H) = \{a \in M \,:\, Ha = H\}$.
$\mathrm{Stab}_r(H)$ acts on H.
$\Sigma_r(H) = \mathrm{Stab}_r(H)$ factored by the kernel of this action.

Proposition

$\Sigma_r(H)$ *is a group acting regularly on* H.

$\Sigma_r(H)$ is called the right Schützenberger group of H.
There is an analogous construction for the left Schützenberger group $\Sigma_l(H)$ of H.

Properties of Schützenberger Groups

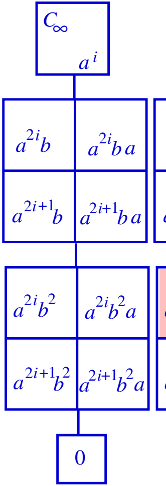
- $\Sigma_r(H) \cong \Sigma_l(H)(= \Sigma(H))$.
- $|\Sigma(H)| = |H|$.
- If $x\mathcal{R}y$ then $\Sigma(H_x) \cong \Sigma(H_y)$.
- If $x\mathcal{L}y$ then $\Sigma(H_x) \cong \Sigma(H_y)$.
- If H contains an idempotent then H is a (maximal sub)group of M and $H \cong \Sigma(H)$.

A monoid is a 'union' of groups, many of which are isomorphic, and some of which are actually in the monoid.

Example

$$
\begin{aligned}
M &= \langle a, a^{-1}, b \mid aa^{-1} = a^{-1}a = 1,\ b^3 = b^2, \\
&\quad a^2 b = b a^2,\ bab = 0 \rangle \\
&= \{a^i, a^i b^j a^k \ :\ i \in \mathbb{Z},\ j = 1, 2, \\
&\quad k = 0, 1\} \cup \{0\}.
\end{aligned}
$$

$\Sigma(H_b) \cong C_\infty$
$\Sigma(H_{b^2}) \cong C_\infty \cong H_{b^2}$.



Cayley Graphs: Monoids vs. Groups

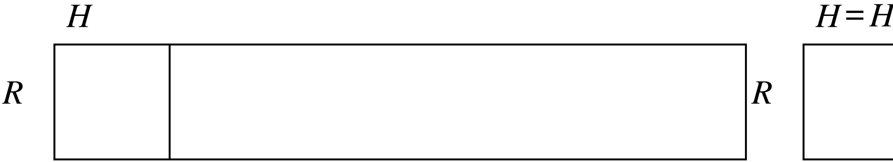
- Cayley graph is ‘essentially’ a monoid concept. (Compare with: transformations/permutations; words/reduced words.)
- The Cayley graph of a general monoid possesses group-theoretic and order-theoretic aspects.
- Main difference between groups and monoids: poset aspect, direction, (ir)reversibility.
- Many concepts/viewpoints of combinatorial/geometrical/computational group theory have natural extensions in the ‘world’ of monoids.

Cayley Graphs: Monoids vs. Groups

- For instance:
  - Word problems, decidability: Markov, Post, Adjan, ...
  - Diagrams, pictures: Remmers, Guba, Howie, Pride, ...
  - Todd–Coxeter, Knuth–Bendix procedures: Neumann, Jura, St Andrews group, ...
  - Automatic structures: St Andrews group, Thomas, Otto, Kambites, Duncan, ...
  - Topology: Stephen, Margolis, Meakin, Steinberg, ...
  - Categories, groupoids: Lawson, Gilbert, ...
  - Ends: Jackson, Kilibarda.
  - Combinatorial theory: the main topic of this talk.
  - ...
- Even if considering Cayley graphs and related topics in this more general setting of monoids may not yield solutions of the hard group-theoretic problems, it may aid understanding of the fundamental concepts involved.

Generators

M – a monoid; H – an $\mathcal{H}$-class; R – the $\mathcal{R}$-class of H.



H_i $(i \in I)$ – the $\mathcal{H}$-classes in R $(H_1 = H)$.
$r_i \in M$ $(i \in I)$ – 'representatives': $Hr_i = H_i$.
r_i' $(i \in I)$ – their 'inverses': $hr_ir_i' = h$ $(h \in H)$.

Theorem

If M is generated by A then the Schützenberger group $\Sigma(H)$ is generated by the elements r_iar_j' $(i, j \in I,\ a \in A,\ Hr_ia = H_j)$.

Remark

Observe a close analogy with the Schreier generating set for a

Finite Generation (1)

Definition

The index of an $\mathcal{H}$-class is the number of $\mathcal{H}$-classes in its $\mathcal{R}$-class.

Corollary

The Schützenberger group of an $\mathcal{H}$-class of finite index in a finitely generated monoid is itself finitely generated.

Corollary

Let M be a monoid with finitely many $\mathcal{H}$-classes (i.e. finitely many left and right ideals). Then M is finitely generated if and only if all its Schützenberger groups are finitely generated.

Finite Generation (2)

Definition

A monoid is regular if for every $x \in M$ there exists $y \in M$ such that $xyx = x$.

Corollary

Let M be a regular monoid with finitely many idempotents. M is finitely generated if and only if all its maximal subgroups are finitely generated.

Presentations: Maximal Subgroups

Theorem

A maximal subgroup of finite index in a finitely presented monoid is itself finitely presented.

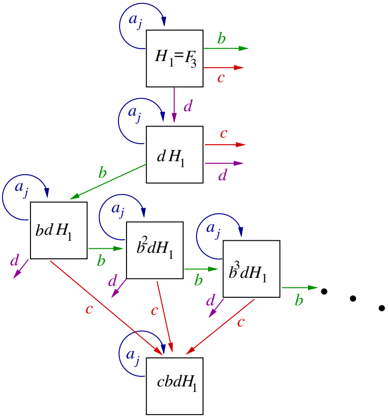
Proof is a straightforward modification of the Reidemeister–Schreier Theorem for subgroups of groups, building on the Schreier-type generating set from above.

Corollary

A regular monoid with finitely many idempotents is finitely presented if and only if all its maximal subgroups are finitely presented.

Presentations: Schützenberger Groups – an Example

G – the group $\langle a_1, a_2, a_3, a_4 \mid a_1 a_2 = a_3 a_4 \rangle$ (free of rank 3).
$M = \langle G, b, c, d \mid a_j b = b a_j^2,\ cb^2 = cb,\ a_j d = d a_j,\ cbda_j = a_j cbd \rangle$.
The left Cayley graph between 1 and $H = H_{cbd}$:



Theorem

H has index 1;
its Schützenberger group is

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i}\ (i = 0, 1, 2, \ldots) \rangle.$$

Presentations: Schützenberger Groups

Theorem

Let M be a monoid with finitely many left and right ideals. Then M is finitely presented if and only if all its Schützenberger groups are finitely presented.

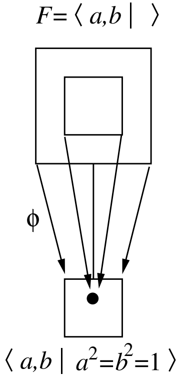
Other Properties

Theorem (Gray, NR)

Let M be a monoid with finitely many left and right ideals. Then M is residually finite if and only if all its Schützenberger groups are residually finite.

Example (Campbell, Robertson, Thomas, NR)

There exists a monoid which is a disjoint union of two copies of the free group of rank 2 which is not automatic.



Question

Suppose M is a monoid with finitely many left and right ideals. If M is automatic, are all its Schützenberger groups automatic? If M is hyperbolic, are all its Schützenberger groups hyperbolic?

Finite Complement (Rees Index)

Theorem

Let M be a monoid, and let $N \leq M$ be such that $M \setminus N$ is finite. Then M is finitely generated (resp. finitely presented) if and only if N is finitely generated (resp. finitely presented).

Remarks

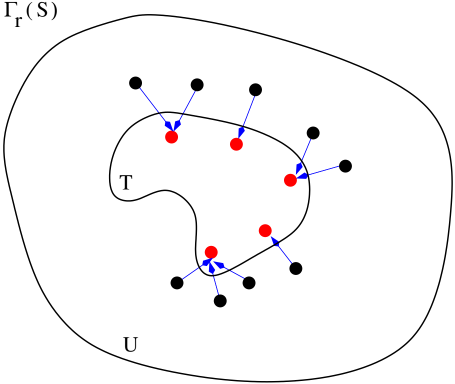
- Analogous to Schreier/ Reidemeister–Schreier theorems for groups.
- Proofs conceptually similar, but very different in details.
- Proof of finite presentability surprisingly hard.
- Analogous theorems for many other finiteness conditions, including residual finiteness (Thomas, NR), automaticity (Hoffmann, Thomas, NR), and many conditions to do with ideals (Malcev, Mitchell, NR).
- Open: finite complete rewriting systems, FDT.

Boundaries

Joint work with R. Gray.

$M = \langle A \rangle$ – a monoid; $N \leq M$.

The right boundary of N: Elements of N which, considered as points in $\Gamma(M, A)$ receive an edge from the outside.

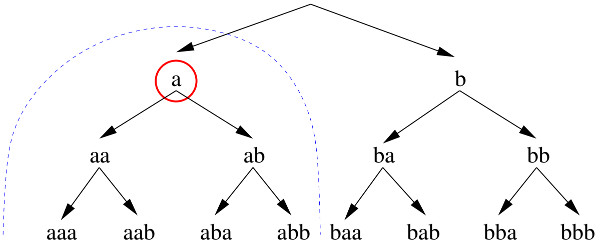


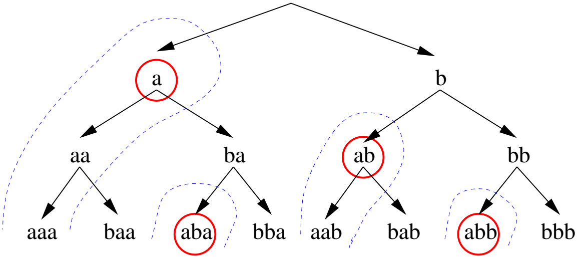
Boundary := left boundary $\cup$ right boundary.

Boundaries: Example

$S = \{a, b\}^*$, $T = aS$.

Right boundary: Left boundary:





Finite Boundaries

Proposition

$N \leq M$ has a finite boundary if any of the following hold:

- *$M \setminus N$ is finite;*
- *$M \setminus N$ is an ideal;*
- *N is an ideal, $M \setminus N$ is a submonoid and the orbits of N under the action of $M \setminus N$ are finite.*

Proposition

Having a finite boundary is independent of the choice of (finite) generating set.

Finite Generation and Presentability

Theorem

Let M be a finitely generated monoid, and let $N \leq M$ have finite boundary.

- *N is finitely generated.*
- *If M is finitely presented then N is finitely presented.*

Questions

How about other properties: FDT, automaticity, etc.?

One-sided Boundaries

Theorem

Let F be a finitely generated free monoid, and let $N \leq F$. If N is finitely generated and has a finite right boundary, then N is finitely presented.

Theorem

Let $M = \bigcup_{i \in I} S_i$ be a finite disjoint union of subsemigroups. If each S_i is finitely presented and has a finite boundary in M then M is finitely presented as well.

A Strange Union

Proposition (Araujo, NR)

If S is the four element semigroup

S	a	b	c	0
a	a	a	c	0
b	b	b	c	0
c	0	0	0	0
0	0	0	0	0

then $S \times C_\infty$ is not finitely presented, but is a disjoint union of two finitely presented subsemigroups $\{a\} \times C_\infty (\cong C_\infty)$ and $\{b, c, 0\} \times C_\infty$.

Green Index: a Preview

- Boundaries do not bring the Rees and group indices closer together.
- A new development: Green index. (Also with R. Gray.)
- Unifies group index, Rees index and Schützenberger groups.
- For finite presentability it features a curious two-directional Reidemeister–Schreier type rewriting.

Project: Monoids, Groups, Posets, Actions

Haven't mentioned: M acts on the set of $\mathcal{R}$-classes, and on the set of $\mathcal{L}$-classes (on the left and right respectively :-( ).
One would like to prove results of the following type for some properties $\mathcal{P}$:
A monoid M satisfies $\mathcal{P}$ if and only if:

- All Schützenberger groups satisfy $\mathcal{P}$;
- The actions of M on $M/\mathcal{R}$ and $M/\mathcal{L}$ satisfy a related property $\mathcal{P}'$;
- The posets $M/\mathcal{R}$ and $M/\mathcal{L}$ satisfy another related property $\mathcal{P}''$.

A promising candidate for $\mathcal{P}$: residual finiteness.