Decidability of the WQO problem for permutations under the consecutive pattern involvement

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Introduction: well ordering for posets

- Well order: a totally ordered set with no infinite descending chains – well ordering principle, ordinals, ...
- Partial well order: no infinite descending chains and no infinite antichains.
- Alternative term: well quasi order – WQO for short.
- Cherlin (2011): ‘tame’ (WQO) vs ‘wild’ (non-WQO).
Substructure orderings in combinatorics

- WQO in combinatorics usually arises in connection with substructure orderings (subgraph, induced subgraph, subpermutation, etc.)
- Automatically no infinite descending antichains (size).
- WQO = no infinite antichains.
Famous example: graph minors

Theorem (Robertson, Seymour)

*The set of all finite graphs under the minor ordering is WQO.*

However, under subgraph ordering and induced subgraph ordering there are antichains; e.g.: cycles $C_n$, $n = 3, 4, \ldots$.
Informally: If the entire class is not WQO, can it be algorithmically decided which downward closed subclasses are WQO?

Problem
Given a class $\mathcal{C}$ of combinatorial objects and a partial ordering on $\mathcal{C}$ is the following algorithmic problem decidable?

- **INPUT**: A finite collection $S_1, \ldots, S_m$ of structures from $\mathcal{C}$, which define a downward closed class

$$\mathcal{D} = \text{Av}(S_1, \ldots, S_m) = \{ S \in \mathcal{C} : S_i \not\leq S \text{ for all } i = 1, \ldots, m \}.$$ 

- **OUTPUT**: YES if $\mathcal{D}$ is WQO, NO if $\mathcal{D}$ is not WQO.
Example: subgraph ordering

Theorem (Ding 1992)

A downward closed set of graphs under the subgraph relation is WQO iff it contains only finitely many cycles and double-ended forks.

Corollary

The WQO problem is decidable for graphs under the subgraph relation.

HOWEVER: the problem is OPEN for the induced subgraph relation (Lozin et al.), digraphs, tournaments (Cherlin & Latka), ...

University of St Andrews Nik Ruškuc: Deciding WQO
Words: subword ordering

$A$ – a finite alphabet; $A^*$ – all words over $A$.

(Scattered) subword ordering:

$x_1 x_2 \ldots x_m \leq y_1 y_2 \ldots y_n \iff x_1 x_2 \ldots x_m = y_{i_1} y_{i_2} \ldots y_{i_m}$ for some $1 \leq i_1 < i_2 < \cdots < y_m \leq n$.

Example: $aaa \leq ababa$, $bbb \not\leq ababa$.

**Theorem (Higman 1952)**

$A^*$ is WQO under the subword ordering.

This (and/or Kruskal’s Tree Theorem) underpin all non-trivial WQO results.
Words: factor ordering

Factor (or contiguous subword) ordering on $A^*$:

$$x_1 x_2 \ldots x_m \leq y_1 y_2 \ldots y_n \iff x_1 x_2 \ldots x_m = y_i y_{i+1} \ldots y_{i+m-1}$$

for some $i$.

Example: $aaa \not\leq ababa$, $bab \leq ababa$. 
Given: \( C = \text{Av}(w_1, \ldots, w_m) \) – a downward closed set under factor ordering.

Note: \( C \) is a regular language.

Define a directed graph \( \Gamma(C) \) as follows.

Let \( \ell = \max\{|w_1|, \ldots, |w_m|\} \).

Vertices: \( C \cap A^\ell \).

Edges: \( a_1a_2\ldots a_\ell \rightarrow a_2\ldots a_\ell a_{\ell+1} \).

Facts

- Every word \( w \in C \) with \( |w| \geq \ell \) defines a path in \( \Gamma(C) \).
- Every path in \( \Gamma(C) \) defines a unique word \( w \in C \) with \( |w| \geq \ell \).
WQO problem for factor ordering (2)

Definition
A directed cycle is said to be an in-out cycle if it contains a vertex of indegree $> 1$ and a vertex of out-degree $> 1$.

Fact
In-out cycles in $\Gamma(C)$ lead to antichains.

Example
Let: $C = \text{Av}(baa, bab)$.

An in-out cycle: $bbb \rightarrow bbb$.
Antichain: $ab^i a$, $i \geq 3$. 
Theorem
\[ \mathcal{C} = \text{Av}(w_1, \ldots, w_m) \text{ contains an antichain if and only if } \Gamma(\mathcal{C}) \text{ contains an in-out cycle.} \]

Corollary
WQO problem is decidable for \( A^* \) under the factor ordering.

This result is a special case of the following:

Theorem (Atminas, Lozin, Moshkov 2013)
It is decidable in polynomial time whether a regular language over \( A \) contains an antichain under the factor ordering.
Permutations

Permutation = a sequence \( \sigma = s_1 \ldots s_n \) s.t.\
\[ \{s_1, \ldots, s_n\} = \{1, \ldots, n\}. \]

\( S \) = the set of all permutations.

\( S_n \) = all permutations of length \( n \); \( S = \bigcup_{n=1}^{\infty} S_n \).

Canonical representatives of non-repeating sequences.

Example

\( \text{perm}(2, 7, 5) = 132 = \begin{array}{cc}
  \cdot & \cdot \\
  \cdot & \cdot \\
\end{array}, \text{perm}(1, e, \pi, i^2) = 2341 = \begin{array}{ccc}
  \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot \\
\end{array}. \)
Permutations: involvement ordering

Analogous to subword ordering.

\[ s_1 \ldots s_m \leq t_1 \ldots t_n \iff s_1 \ldots s_m = \text{perm}(t_{i_1} \ldots t_{i_m}) \text{ for some } 1 \leq i_1 < \cdots < i_m \leq n. \]

Example

\[
\begin{align*}
231 & \leq 3142 \\
\begin{array}{c}
\bullet \\
\bullet
\end{array} & \leq \\
\begin{array}{c}
\bullet \\
\bullet \bullet \bullet
\end{array} & \not\leq \\
123 & \not\leq 3142 \\
\begin{array}{c}
\bullet \\
\bullet
\end{array} & \not\leq \\
\begin{array}{c}
\bullet \\
\bullet \bullet \bullet
\end{array}
\end{align*}
\]

Open Problem

Is the WQO problem decidable for permutations under the involvement ordering?
Permutations: consecutive involvement ordering

\[ s_1s_2 \ldots s_m \leq t_1t_2 \ldots t_n \iff s_1s_2 \ldots s_m = \text{perm}(t_it_{i+1} \ldots t_{i+m-1}) \]
for some \( i \).

Example
\[ 231 \not\leq 3142, \quad 213 \leq 3142. \]

Question
Is the WQO problem decidable for permutations under the consecutive involvement ordering?
Graph $\Gamma(C)$

$C = \text{Av}(\pi_1, \ldots, \pi_m); \ell = \max\{|\pi_1|, \ldots, |\pi_m|\}$.

Vertices: $C \cap S_{\ell};$

Edges: $a_1 \ldots a_{\ell} \rightarrow b_1 \ldots b_{\ell} \iff \text{perm}(a_2 \ldots a_{\ell}) = \text{perm}(b_1 \ldots b_{\ell-1})$.

Facts

- Every permutation $\sigma \in C$ with $|\sigma| \geq \ell$ defines a path in $\Gamma(C)$.
- BUT: a path in $\Gamma(C)$ may correspond to several $\sigma$ (an ambiguous path).
First obstacle to WQO: in-out cycles

Fact

*If* $\Gamma(C)$ *has an in-out cycle then* $C$ *contains an infinite antichain.*

Example

$C = \text{Av}(231, 312, 1234, 1243, 1432, 2431, 3142, 4213, 4321)$. 

![Diagram of $\Gamma(C)$ and an antichain]

Antichain: 

for $k = 1, 2, \ldots$
Bicyclic classes

**Bicycle:** digraph consisting of two simple cycles connected by a single non-trivial path.

\( \mathcal{C} \) is **bicyclic** if \( \Gamma(\mathcal{C}) \) is a bicycle (or a degenerate form, where one or both cycles are not present).

**Fact**

*If \( \mathcal{C} \) has no in-out cycles then it is a finite union of bicyclic classes.*

So we may restrict our WQO considerations to bicyclic classes.
Second obstacle to WQO: ambiguous paths

Fact

If a bicyclic class $\mathcal{C}$ has an ambiguous path which begins and ends on the same cycle then $\mathcal{C}$ contains an infinite antichain.

Example

Antichain: for $k = 1, 2, \ldots$
Going around a cycle

Consider a cycle with no ambiguous paths.

The effect of repeatedly going around the cycle can be viewed as a permutation $\alpha = a_1 \ldots a_n$ which is repeatedly juxtaposed with itself according to a fixed rule.

This in turn can be represented as a juxtaposition $\alpha' \alpha'' = a'_1 \ldots a'_n a''_1 \ldots a''_n$ of two copies of $\alpha$.

Let $a_i, a_j$ be two entries, consecutive in value.

If $a''_i < a'_i < a'_j < a''_j$ we say that $(a_i, a_j)$ is a nested interval of $\alpha$.

Example

Cycle: \begin{tabular}{c}
\begin{tikzpicture}
\draw[fill=blue!20] (0,0) circle (0.1);
\draw[fill=blue!20] (0.5,0) circle (0.1);
\draw[fill=blue!20] (1,0) circle (0.1);
\draw[fill=blue!20] (1.5,0) circle (0.1);
\draw[fill=blue!20] (2,0) circle (0.1);
\draw[fill=blue!20] (2.5,0) circle (0.1);
\end{tikzpicture}
\end{tabular}
Juxtaposition: \begin{tabular}{c}
\begin{tikzpicture}
\draw[fill=blue!20] (0,0) circle (0.1);
\draw[fill=blue!20] (0.5,0) circle (0.1);
\draw[fill=blue!20] (1,0) circle (0.1);
\draw[fill=blue!20] (1.5,0) circle (0.1);
\draw[fill=blue!20] (2,0) circle (0.1);
\draw[fill=blue!20] (2.5,0) circle (0.1);
\end{tikzpicture}
\end{tabular}
\begin{tabular}{c}
\begin{tikzpicture}
\draw[fill=blue!20] (0,0) circle (0.1);
\draw[fill=blue!20] (0.5,0) circle (0.1);
\draw[fill=blue!20] (1,0) circle (0.1);
\draw[fill=blue!20] (1.5,0) circle (0.1);
\draw[fill=blue!20] (2,0) circle (0.1);
\draw[fill=blue!20] (2.5,0) circle (0.1);
\end{tikzpicture}
\end{tabular}
\begin{tabular}{c}
\ldots
\end{tabular}
\end{tabular}

Note: No antichains here.
Third obstacle to WQO: inserting a point into a nested interval

Fact

If a bicyclic class $C$ has an ambiguous path which begins on the initial cycle and ends on the connecting path which allows insertion into a nested interval of $\alpha$ then $C$ contains an infinite antichain.

Example

Antichain: $\ldots$
Theorem (McDevitt, NR)

A downward closed class $C = \text{Av}(\pi_1, \ldots, \pi_m)$ of permutations under the consecutive involvement ordering is WQO iff the following three conditions are satisfied:

1. $\Gamma(C)$ has no in-out cycles;
2. no bicyclic component of $\Gamma(C)$ has an ambiguous path starting and ending on the same cycle;
3. no bicyclic component of $\Gamma(C)$ permits insertion into a nested interval.

Corollary (McDevitt, NR)

WQO problem is decidable for permutations under the factor ordering.
Concluding remarks

Similar techniques, involving the graph $\Gamma(C)$, can be used to prove that the atomicity problem is decidable for: (a) permutations under the consecutive factor ordering; and (b) words under factor ordering.

A downward closed set is atomic if it is not a union of two proper downward closed subsets; equivalently: Joint Embedding Property.

Braunfeld (2019) proved that atomicity is undecidable for: (a) graphs under the induced subgraph ordering; and (b) 3-dimensional permutations under the involvement ordering.
Questions

Are the atomicity and WQO problems decidable for 3-dimensional permutations, where in two dimensions the ordering is consecutive, and in the remaining one it is not? What can be said about higher-dimensional permutations?

Question

To what extent can the WQO and atomicity results be extended to infinitely based classes? E.g. Av($B$) where $B$ is a periodic antichain?
THANK YOU!