On groups of units in one relation monoids and inverse monoids

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Theorem (Adjan 1966)

The word problem for any special one-relator monoid
\( M = \text{Mon}\langle A \mid r = 1 \rangle \) is decidable.

- Adjan’s original proof consists of 38 pages of heavy combinatorics on words.
- Zhang (1992) provided a 3 page proof using rewriting systems.
- The key is to determine the group of units of \( M \).
- This is only implicit in the first Zhang paper, and becomes explicit in the second one, dealing with special monoids in general, also 1992.
The group of units of $\text{Mon} \langle A \mid r = 1 \rangle$

- Decompose $r \equiv r_1r_2\ldots r_k$, where the $r_i$ are minimal subject to being invertible (the invertible pieces of $r$).
- Suppose that the set of words $\{r_1, \ldots, r_k\}$ in fact has size $l \leq k$.
- Introduce new generating symbols $x_1, \ldots, x_l$, one per distinct piece.
- Let $\bar{r}_i$ be the element of $\{x_1, \ldots, x_l\}$ representing $r_i$.


The group of units $U(M)$ of $M = \text{Mon} \langle A \mid r = 1 \rangle$ is defined by the presentation $\text{Gp} \langle x_1, \ldots, x_l \mid \bar{r}_1\bar{r}_2\ldots\bar{r}_k = 1 \rangle$. 
The key fact:
If \( xy \) and \( yz \) are invertible, then so are in fact \( x, y \) and \( z \).

Corollary: The pieces of \( r \) do not overlap.

(Although a piece can contain another piece.)
Let $R$ be the submonoid of right units, and let $u \in R$.

Let $v \in M$ be such that $uv = 1$.

Consider the sequence of applications of $r = 1$ taking $uv$ to 1, and spot the first instance when it is applied at the boundary between $u$ and the rest. This shows that $u$ has a suffix, which is a prefix of $r$.

**Theorem**

$R$ is generated by the prefixes of $r$.

**Theorem**

$R \setminus U(M)$ is an ideal of $R$, and hence $U(M)$ is generated by the invertible prefixes of $r$, and therefore also by the pieces $r_1, \ldots, r_k$. 
Define relation

Consider a word \( w \in \{r_1, \ldots, r_k\}^* \subseteq A^* \) which equals 1 in \( M \), and look at the sequence of applications of \( r = 1 \) transforming it to 1.

These applications cannot take place ‘across’ several \( r_j \) because of the non-overlap condition.

They can happen inside a single \( r_j \) – a small trick (based on Freihetsatz for one-relator groups) is used to show that these ’don’t matter’.

The presentation result follows.
Computing the presentation

Is the presentation for \( U(M) \) algorithmically computable from the presentation \( \text{Mon}\langle A \mid r = 1 \rangle \)?

This boils down to computing the pieces of \( r \).

1. Let \( W := \{ r \} \).

2. For any two (not necessarily distinct) \( r_1, r_2 \in W \), if they can be decomposed as \( r_1 \equiv st, r_2 \equiv tu \), add \( ts \) and \( ut \) to \( W \).

3. Repeat 2, until no new words are added.

At termination, \( W \) consists of certain cyclic conjugates of \( r \).

This corresponds to a decomposition \( r \equiv r_1 r_2 \ldots r_k \).

**Theorem**

\( r_1, \ldots, r_k \) are precisely the minimal invertible pieces of \( r \).
From another viewpoint...

- Given words $r = r(y_1, \ldots, y_l)$, and $s_i \in A^*$ ($i = 1, \ldots, l$) it is decidable whether the $r_i$ are the pieces of $\text{Mon}\langle A \mid r(s_1, \ldots, s_l) = 1 \rangle$.
- It is decidable whether $\text{Mon}\langle A \mid r = 1 \rangle$ is a group.
Special inverse monoids $M = \text{Inv}\langle A \mid r = 1 \rangle$

**Theorem (Ivanov, Margolis, Meakin 2001)**

*If the word problem is decidable for special inverse monoids $\text{Inv}\langle A \mid r = 1 \rangle$, then the word problem is also decidable for every one-relator monoid.*

- $r$ can further be assumed to be reduced.
- A body of work by MM and collaborators considering different types of relator $r$ (e.g. idempotent, strictly positive, etc.)
Generators for $U(M)$

Throughout $M := \text{Inv}\langle A \mid r = 1 \rangle$.

**Theorem (IMM 2001)**

$U(M) = \langle r_1, \ldots, r_k \rangle$ where $r \equiv r_1 r_2 \ldots r_k$ is the decomposition of $r$ into invertible pieces.

- The proof is very similar to the monoid case.
- Right units = the Schützenberger graph of 1.
- Stephen’s procedure implies that the right units are generated by the prefixes of $r$.

**Question**

Is $U(M)$ defined by $\text{Gp}\langle x_1, \ldots, x_l \mid \bar{r}_1 \bar{r}_2 \ldots \bar{r}_k = 1 \rangle$, as in the monoid case?
Example: O’Hare monoid

\[ M = \text{Inv}\langle a, b, c, d \mid abcdacdadabbcda cd = 1 \rangle. \]

(Margolis, Meakin, Stephen 1997)

The words \( abcd \), \( acd \), \( ad \) and \( abbcda \) are all invertible in \( M \).
\[ M = \text{Inv}\langle a, b, c, d \mid abcdacdadabbcda \rangle = 1 \rangle. \]

The words \( abcd, acd, ad \) and \( abbcda \) are all invertible in \( M \).

If a proper subword is invertible then all \( a, b, c, d \) are invertible (i.e. \( M \) is a group).

But: the bicyclic monoid
\[ B = \text{Mon}\langle x, y \mid xy = 1 \rangle = \text{Inv}\langle x \mid xx^{-1} = 1 \rangle \] is a homomorphic image of \( M \) via \( a \mapsto x, b, c \mapsto 1, d \mapsto y \).

Hence \( abcd, acd, ad \) and \( abbcda \) are the pieces of \( r \).

**Question**

Is \( U(M) \) defined by \( \text{Gp}\langle x, y, z, t \mid xyzty = 1 \rangle \)?
\[ M = \text{Inv}\langle a, b, c, d \mid abcd \cdot acd \cdot ad \cdot abbcd \cdot acd = 1 \rangle \]

\( M \text{ naturally maps onto} \)

\[ G = \text{Gp}\langle a, b, c, d \mid abcdadadabbdcdacd = 1 \rangle \]
\[ = \text{Gp}\langle a, b, c, y \mid abcyybbcycya^{-1} = 1 \rangle \quad (y = dc) \]
\[ = \text{Gp}\langle a \mid \rangle \ast \text{Gp}\langle b, c, y \mid bcycyybbcycy = 1 \rangle \]

Where does \( U(M) \) go under this homomorphism?

\[ \langle abcd, acd, ad, abbcd \rangle \]
\[ = \langle abcya^{-1}, acya^{-1}, aya^{-1}, abbcya^{-1} \rangle \]
\[ \cong \langle bcyc, cy, y, bbcy \rangle \]
\[ = \langle b, c, y \rangle \]
\[ = \text{Gp}\langle b, c, y \mid bcycyybbcycy = 1 \rangle. \]
On the other hand . . .

\[ M = \text{Inv}\langle a, b, c, d \mid abcd \cdot acd \cdot ad \cdot abbcd \cdot acd = 1 \rangle \]

**Lemma**

*In an inverse semigroup \( S \), if \( uvRu \) then \( uvv^{-1} = u \).*

What happens if in \( M \) we substitute \( b \rightarrow aba^{-1}, c \rightarrow aca^{-1}, \)
\( y \rightarrow aya^{-1} = adaa^{-1} = ad \) into \( bcycybbbcycy \)?

\[
aba^{-1} \cdot aca^{-1} \cdot ad \cdot aca^{-1} \cdot ad \cdot ad \cdot aba^{-1} \cdot aba^{-1} \\
\cdot aca^{-1} \cdot ad \cdot aca^{-1} \cdot ad \\
= abcdacdadabbcdaacd = 1.
\]

So, \( U(M) \) satisfies the relation \( bcycybbbcycy = 1 \)!!!

Hence \( U(M) = \text{Gp}\langle b, c, y \mid bcycybbbcycy \rangle \).
An isomorphism theorem

Let $M = \text{Inv} \langle A \mid r = 1 \rangle$ be a special inverse one-relator monoid.

Let $r \equiv r_1 r_2 \ldots r_k$ ($r_i$ the pieces).

Let $s_1, \ldots, s_l$ be free generators for the subgroup $\langle r_1, \ldots, r_k \rangle \leq \text{FG}_A$.

Let $y_1, \ldots, y_l$ be new generators, representing $s_1, \ldots, s_l$.

Express each $r_i$ in terms of the $y_j$, substitute into $r$, and freely reduce, yielding $r' = r'(y_1, \ldots, y_l)$.

**Theorem (R. Gray, NR)**

*If the subgroup of $\text{Gp} \langle A \mid R \rangle$ generated by $s_1, \ldots, s_l$ is defined by $\text{Gp} \langle y_1, \ldots, y_l \mid r' = 1 \rangle$ then the group of units of $M$ is also defined by the same presentation.*
A question about one-relator groups

Question
Suppose $u_1, \ldots, u_m \in FG_A$ are a basis for $\langle u_1, \ldots, u_m \rangle \leq FG_A$. Let $r = r(y_1, \ldots, y_m)$ be a freely reduced word. Under what conditions is the subgroup of the one-relator group

$$Gp\langle A \mid r(u_1, \ldots, u_m) = 1 \rangle$$

generated by $u_1, \ldots, u_m$ isomorphic to

$$Gp\langle y_1, \ldots, y_m \mid r = 1 \rangle$$?
Question

When is \( \langle u_1, \ldots, u_m \rangle \leq \text{Gp}\langle A \mid r(u_1, \ldots, u_m) = 1 \rangle \) isomorphic to \( \text{Gp}\langle y_1, \ldots, y_m \mid r = 1 \rangle \)?

- One instance is when each \( u_i \) contains a letter which is not contained in any other \( u_j \). (RG & NR)

- Generalisation of Corollary 4.10.2 in Magnus, Karrass, Solitar:
  The subgroup of \( \text{Gp}\langle x, c, \ldots, t \mid R(x^\gamma, c, \ldots, t) \rangle \) generated by \( x^\gamma, c, \ldots, t \) has the presentation \( \text{Gp}\langle b, c, \ldots, t \mid R(b, c, \ldots, t) \rangle \).

- Would Nielsen reduced help? How?

- Or something stronger than Nielsen reduced? What?
  (Presumably, somehow reflecting the non-overlapping of pieces)?

- Is there a (combinatorial?) criterion for \( u_1, \ldots, u_m \) to be a basis of \( \langle u_1, \ldots, u_m \rangle \)?
Another link with one-relator groups

**Theorem**

*If* $r$ *is cyclically reduced, and* $\text{Gp}\langle A \mid r = 1 \rangle$ *is coherent then the group of units of* $\text{Inv}\langle A \mid r = 1 \rangle$ *is finitely presented.*

**Conjecture (G. Baumslag 1974)**

Every one-relator group is coherent.
Computing the pieces

- It is still the case that the pieces of the relator must not overlap (because of minimality).
- However, this is not a characterising property any longer.
- For instance, in

\[ M = \text{Inv} \langle a, b, c, d \mid abcdacdadabbcdacd = 1 \rangle \]

the relator has no self-overlaps!

**Question**
Is there an algorithm which given a presentation \( \text{Inv} \langle A \mid r = 1 \rangle \) computes the pieces of \( r \)?

**Question**
Is it decidable whether \( \text{Inv} \langle A \mid r = 1 \rangle \) is a group?
Should the notion of overlap be extended/modified to include the cases when apparently there are no overlaps? In what sense does O’Hare relator overlap with itself?

(R. Gray algorithm) Let $R$ be the set of all prefixes of $r$ (in the Munn tree sense). Let $P$ be the submonoid of $FG_A$ generated by $R$. Membership in $P$ is decidable (by an automaton; Benois 1969). Is $\{w \in R : w^{-1} \in P\}$ the set of invertible prefixes of $r$? [If we could compute the invertible prefixes, then we would immediately get the pieces as well.]

Apart from being interesting in itself, resolving the question of computing the pieces, presumably accompanied by some kind of characterisation of what it means to be a piece, is likely to provide a key (missing) component in determining the presentation.
In the case of special monoids, Otto and Zhang (1991) prove:

**Theorem**

Let \( M = \text{Mon} \langle A \mid r = 1 \rangle \). Every word \( u \in A^* \) can be uniquely decomposed as \( u_0 a_1 u_1 a_2 \ldots a_m u_m \), where \( a_1, \ldots, a_m \in A \) and \( u_0, \ldots, u_m \) are maximal invertible factors. If \( v = v_0 a'_1 v_1 a'_2 \ldots a'_n v_n \) decomposed in the same way, we have \( u = v \) in \( M \) iff \( m = n \), \( a_i = a'_i \) and \( u_j = v_j \) for all \( i, j \).

**Question**

What would an analogue for inverse monoids look like? Presumably, words would be replaced by Munn trees? Maximal invertible prefixes?
THANK YOU FOR LISTENING!