Pattern Avoidance in Permutations

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Sorting With a Stack

Consider the set of all finite permutations that can be sorted by a stack.

**Proposition.** A permutation $\pi = \pi_1\pi_2\ldots\pi_n$ can be sorted by a stack if there do not exist $i, j, k \in \{1, \ldots, n\}$ such that $i < j < k$ and $\pi_k < \pi_i < \pi_j$.

**Proposition.** The number of permutations of length $n$ which can be sorted by a stack is equal to the $n$th Catalan number $C_n$. 
Pattern Involvement

Let us have two sequences: \( \sigma = \sigma_1\sigma_2\ldots\sigma_m \) and \( \tau = \tau_1\tau_2\ldots\tau_n \) of (say) positive integers.

- \( \sigma \) and \( \tau \) are order isomorphic (\( \sigma \cong \tau \)) if \( m = n \) and for all \( i, j = 1, \ldots, n \) we have
  \[
  \sigma_i \leq \sigma_j \iff \tau_i \leq \tau_j.
  \]

- \( \sigma \) is involved in \( \tau \) (\( \sigma \preceq \tau \)) if \( \tau \) contains a subsequence \( \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_m} \) \( (i_1 < \ldots < i_m) \) order isomorphic to \( \sigma \).

- If \( \sigma \) is not involved in \( \tau \) we say that \( \tau \) avoids \( \sigma \).

Example. \( 231 \preceq 142635 \). \( 231 \not\preceq 142365 \).
**Closed Classes: Definition**

**Notation.** $S_n$ – the set of all permutations on $\{1, \ldots, n\} = [n]$ (written as sequences of images).

$$S = \bigcup_{i=0}^{\infty} S_n.$$  

**Proposition.** The following two conditions are equivalent for a class (set) $C \subseteq S$ of permutations:

(i) $\pi \in C$ & $\rho \preceq \sigma \Rightarrow \rho \in C$.

(ii) $\sigma \in S \setminus C$ & $\sigma \preceq \tau \Rightarrow \tau \in S \setminus C$.

(iii) There exists a (unique) antichain $B$ of permutations such that $\sigma \in C$ iff $\sigma$ avoids all permutations in $B$.

**Definition.** If $C$ satisfies one (and hence all) of the above conditions we say that $C$ is a closed class. The set $B$ described in (iii) is called the basis for $C$. 


Closed Classes: Examples

Example. The class of permutations sorted by a stack is closed. Its basis is \{231\}.

Example. Permutations obtained by splitting 12\ldots n into two subsequences and interleaving them arbitrarily. For instance: 21356487 \in \mathcal{C}, 321 \notin \mathcal{C}. The basis of this class is \{321\}.

Example. The riffle-shuffle class \mathcal{R}: all the permutations obtained by splitting 12\ldots n into two subsequences 12\ldots i and i + 1,\ldots, n and interleaving them. For instance: 12673485 \in \mathcal{R}, 2143, 2413 \notin \mathcal{R}. The basis of this class is \{321,2143,2413\}. 
Closed Classes: General Questions

Given a closed class $\mathcal{C}$ (by its basis, sorting mechanism, generating mechanism, etc.), we ask the following questions:

**Basis Problem.** Determine the basis of $\mathcal{C}$. Failing this, is the basis finite? What is the number of basis permutations of length $n$ ($n = 1, 2, 3, \ldots$).

**Membership Problem.** Does there exist an algorithm which for every $\pi \in \mathcal{S}$ determines whether or not $\pi \in \mathcal{C}$? Is there an efficient (linear, polynomial, etc.) algorithm for doing this?

**Enumeration Problem.** Determine the numbers $e_n(\mathcal{C}) = |\mathcal{C} \cap \mathcal{S}_n|$ of all permutations in $\mathcal{C}$ of length $n$. 

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Remarks

• Not all closed classes are finitely based, because there exist infinite antichains of permutations. For example the permutations:

\[(2, 3), 5, 1, 7, 4, 9, 6, \ldots, 2i - 1, 2i + 2, \ldots,\]
\[\ldots 2n + 1, 2n - 2, (2n + 2, 2n + 3), 2n\]

\[(n = 2, 3, 4, \ldots)\text{ form such an antichain (Spielman, Bona).}\]

• Finite basis implies a polynomial membership testing algorithm.
Enumeration

• If $C$ is a class defined by a single basis permutation of length 3 then $e_n(C) = C_n$.

• Gessel: a formula for the enumeration of the class with basis $\{1234\}$.

• Bona: basis $\{1342\}$.

• Regev: an asymptotic formula for basis $\{123\ldots n\}$.

• Simion, Schmidt: all basis permutations of length 3.

• West: one basis permutation of length 3, and one of length 4.
Stanley–Wilf Conjecture

**Conjecture.** (Stanley–Wilf) For every closed class $C$ there exists $q$ such that $e_n(C) \leq q^n$.

**Remark.** It is sufficient to consider classes with bases of size one.

**Theorem.** (Noga, Friedgut) For $e_n(C) \leq q^n\gamma(n)$ where $\gamma$ is a function that increases very (very) slowly.

**Conjecture.** (Gessell) The enumeration sequence of every finitely based closed class satisfies a recursive formula with polynomial coefficients.

**Remark.** Not true for infinitely based closed classes (Atkinson, Murphy).
Two Stacks in Series

$S^2$: permutations that can be sorted with two stacks connected in series.

**Example.** $231 \in S^2$, $2435761 \notin S^2$.

**Fact.** $S^2$ is not finitely based.

**Questions.** Find the enumeration sequences for $S^2$ and its basis.

**Conjecture.** The membership problem for $S^2$ is NP-complete (Atkinson). The membership problem for $S^2$ is polynomial (Murphy).

* Delete as appropriate.
Dunedin–St Andrews Collaboration

- M.D. Atkinson, M.M. Murphy, N. Ruskuc: two ordered stacks in series: basis and enumeration.
- M.D. Atkinson, M.M. Murphy, N. Ruskuc: partially well ordered closed classes; the class of one-stack sortable permutations is p.w.o.
- M. Albert, M.D. Atkinson, N. Ruskuc: closed classes and languages.
- M.M. Murphy: extensive structure theory, new methods for defining closed classes, beginnings of a theory of antichains.
Bounded Classes

Suppose that we are generating permutations using a mechanism with finite memory of size $M$.

If the system is full it must output before taking more input. So we cannot generate any permutation of the form

$$(M + 1, t_1, \ldots, t_M), (t_1, \ldots, t_M < M + 1). \quad (*)$$

Let $\Omega_M$ be the class with basis consisting of all permutations of the form $(*)$.

Every closed subclass of $\Omega_M$ is called bounded.
Graphs as Generators

$S_{2,2}$ – two stacks of size 2 in series:

A closed class generated by a graph is called a graph class.
Regular Languages

Let $A$ be a finite alphabet.

A language is any set of (finite) words over $A$.

A language is regular if it is accepted by an automaton.

Regular expressions: well-formed expressions over $A \cup \{\,(,)\,,\,*\,,\,+\,,\,\cdot\,,\,\cap\,,\,-\}\,$.

**Theorem. (Kleene)** A language is regular if and only if it is defined by a regular expression.

**Example.** The language accepted by the automaton $M$ is defined by the regular expression $(a + b)^* - (a + b)^* a a (a + b)^*$.
Regular Languages: Membership and Enumeration

**Fact.** For every regular language $L$ there exists an algorithm which tests the membership in $L$ in linear time.

**Proof.** Feeding the word through an automaton accepting $L$.

For a language $L$ let $e_n(L) = |L \cap A^n|$ be the number of words in $L$ of length $n$.

**Fact.** If $L$ is regular then $e_n(L)$ satisfy a linear recurrence with constant coefficients.
Regular Languages: Membership and Enumeration

Example. Consider the language accepted by the automaton $M$. Let $L^{(i)}$ be the language accepted by $M$ with $s_i$ as the only accept state. Then:

\[
L_0^{(0)} = 1, \quad L_0^{(1)} = 0, \quad L_0^{(2)} = 0,
\]
\[
L_{n+1}^{(0)} = L_n^{(0)} + L_n^{(1)},
\]
\[
L_{n+1}^{(1)} = L_n^{(0)},
\]
\[
L_{n+1}^{(2)} = L_n^{(1)} + 2L_n^{(2)}.
\]

Hence

\[
L_{n+1} = L_{n+1}^{(0)} + L_{n+1}^{(1)} = 2L_n^{(0)} + L_n^{(1)} = L_n + L_{n-1}.
\]
Encoding $\Omega_M$

For $\pi = \pi_1 \pi_2 \ldots \pi_n \in S$ let

$$E(\pi) = p_1 p_2 \ldots p_n$$

where

$$p_i = |\{ j : j \geq i, \pi_j \leq \pi_i \}|$$

(the rank of $\pi_i$ among $\{\pi_i, \ldots, \pi_n\}$). For example

$$E(24163875) = 23131321.$$ 

**Fact.** If $A \subseteq \Omega_M$ then $E(A) \subseteq [M]^*$. 
**Regular Classes**

**Definition.** A set $A \subseteq \Omega_M$ is regular if $E(A)$ is a regular language over $[M]^*$.  

**Fact.** $\Omega_M$ itself is regular.  

**Proof.** $E(\Omega_M) = [M]^* - [M]^*F$, where $F$ is a finite set of words.

**Fact.** Not every subclass of $\Omega_M$ is regular.  

**Proof.** $\Omega_M$ is not partially well ordered, and hence it has uncountably many closed subclasses (Atkinson, Murphy, Ruskuc).
Examples

Example. Let $\mathcal{X}$ be the closed class with basis $\{312, 321, 231\}$. 312 and 321 ensure that $\mathcal{X} \subseteq \Omega_2$. 231 then implies that the subword 22 never appears in the encoding. Thus $E(\mathcal{X}) = E(\Omega_2) - (1 + 2)^*22(1 + 2)^*$.

Theorem. (Atkinson, Livsey, Tuley) Every graph class is regular.

Proof. Construct an automaton. Here are some states and transitions for $S_{2,2}$. 
Deleting/Inserting Elements

Lemma. (Albert, Atkinson, Ruskuc) For any $A \subseteq \Omega_M$ let:

- $A_d$ be the set of all permutations obtained by deleting a single entry in a permutation from $A$;
- $A_D$ be the set of all permutations obtained by deleting any number of entries in a permutation from $A$;
- $A_i$ be the set of all permutations obtained by inserting a single entry in a permutation from $A$;
- $A_I$ be the set of all permutations obtained by inserting any number of entries in a permutation from $A$.

If $A$ is regular then so are $A_d, A_D, A_i$ and $A_I$. 
Regular Classes: General Results

(Albert, Atkinson, Ruskuc)

**Theorem 1.** There is an algorithm which decides whether or not a given regular set $L \subseteq [M]^*$ is the encoding of a (regular) closed class in $\Omega_M$.

**Theorem 2.** A closed bounded class $C$ is regular if and only if its basis $B$ is regular. Moreover, there are algorithms which compute $E(B)$ from $E(C)$ and vice versa.
Regular Classes: General Results

**Corollary 2A.** There is an algorithm which decides whether or not a given closed, bounded class is finitely based or not.

**Facts.** (I) The membership problem in any regular bounded class is decidable in linear time.
(II) The enumeration sequence of a regular bounded class satisfies a linear recurrence with constant coefficients.
Natural Classes

Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be a permutation.

Let $\mathcal{X}$ be the set of all finite permutations involved in $\pi$. Clearly $\mathcal{X}$ is closed.

Closed classes obtained in this way are called natural.

Example. If $\pi = (2, 1, 3, 4, 5, 6, 7, \ldots)$ then $\mathcal{X}$ consists of all permutations of the forms $123\ldots n$ and $2134\ldots n$. 
**Sum Complete Classes**

For permutations $\alpha = \alpha_1 \ldots \alpha_m$ and $\beta = \beta_1 \ldots \beta_n$, define

$$\alpha \oplus \beta = \alpha_1 \ldots \alpha_m \beta'_1 \ldots \beta'_n,$$

where $\beta'_i = \beta_i + m$.

A closed class $\mathcal{X}$ is said to be **sum complete** if

$$\alpha \beta \in \mathcal{X} \Rightarrow \alpha \oplus \beta \in \mathcal{X}.$$

**Examples.** All the following classes are sum complete: one stack sortable, two stacks in series sortable, graph classes, etc. The natural class $\mathcal{X}$ defined by $\pi = (2, 1, 3, 4, \ldots)$ is not sum complete, since $21 \in \mathcal{X}$ but $21 \oplus 21 = 2143 \notin \mathcal{X}$.

**Proposition.** (Atkinson) Every sum complete class is natural.

**Proposition.** (Atkinson) A closed class is sum complete if and only if all its basis permutations are sum indecomposable.
Murphy’s Alternative

**Theorem.** (Murphy) If $\mathcal{X}$ is a natural class defined by a permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$, then at least one of the following two statements is true:

1. $\mathcal{X} = \mathcal{F} \oplus \mathcal{C}$ where $\mathcal{F}$ is finite and $\mathcal{C}$ is sum complete.

2. $\pi$ is periodic and (hence) $\mathcal{X}$ is regular.
Deciding the Join Property

**Theorem.** There is an algorithm which determines for any finitely based class whether or not it is natural.

By taking $\pi : \mathbb{Z} \to \mathbb{Z}$ we can define integral classes.

**Question.** Is there an algorithm which decides whether or not a finitely based class is integral?

A closed class $C$ is said to have the join property if for every $\alpha, \beta \in C$ there exists $\gamma \in C$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

**Fact.** Every natural (resp. integral) class has the join property.

**Question.** Is there an algorithm which decides for any finitely based closed class whether or not it has the join property?