Decidability Questions for Pattern
Avoidance Classes of Permutations

Nik Ruškuc, St Andrews

nik@mcs.st-and.ac.uk
http://www-groups.mcs.st-and.ac.uk/~nik

Cambridge, October, 2004
Template Question

Does there exist an algorithm which does the following:

**INPUT:** A (finite) (def)
defining a pattern avoidance class \( X \) of permutations.

**OUTPUT:** TRUE if \( X \) is (property) , FALSE otherwise?

If such an algorithm exists we say that the property is decidable; if not, it is undecidable
Algorithms

Algorithm

= Turing machine
= effective computation
= computer program
= unambiguous recipe

• know one when you see it;
• approach borrowed from algebra;
• a fresh look at the area;
• ignore technical issues: complexity, efficiency, etc (for the moment);
• computational tools would be useful nevertheless.
Permutations as Patterns

Sequence = a finite list of distinct numbers.

Permutation = a sequence of numbers 1, \ldots, n of length n.

Order isomorphism: for sequences $\sigma = s_1 \ldots s_m$, $\tau = t_1 \ldots t_n$ we say

$$\sigma \leq \tau \iff m = n \& (\forall i, j : s_i \leq s_j \iff t_i \leq t_j).$$

**Example.** $8463 \leq 4231$, $8463 \not\leq 4321$.

**Fact.** For every sequence $\sigma$ there is a unique permutation $\bar{\sigma}$ such that $\sigma \leq \bar{\sigma}$.

**Example.** $\overline{8463} = 4231$. 
Involvement and Avoidance

**Definition.** $\sigma = s_1 \ldots s_m$ is involved in $\tau = t_1 \ldots t_n$ if $\tau$ contains a subsequence order isomorphic to $\sigma$:

$$\sigma \preceq \tau \iff \exists 1 \leq i_1 < i_2 < \ldots < i_m \leq n : t_{i_1} \ldots t_{i_m} \cong \sigma.$$ 

If $\sigma \not\preceq \tau$ we say that $\tau$ avoids $\sigma$.

**Example.** $123 \preceq 32415$, $123 \not\preceq 35421$.

**Fact.** $\preceq$ is a preorder (RT) on the set $T$ of all sequences, and is a (partial) order (RST) on the set $S$ of all permutations.

**Fact.** $(T, \preceq) / \{(\sigma, \tau) : \sigma \preceq \tau \& \tau \preceq \sigma \} \cong (S, \preceq)$. 
Closed (Pattern Avoidance) Classes

**Definition.** A set $X \subseteq S$ of permutations is said to be closed if

$$\sigma \in X \& \tau \preceq \sigma \Rightarrow \tau \in X.$$  

Another name: pattern avoidance class (why: see later).

**Examples.**

- $S$;
- $I = \{1, 12, 123, \ldots\}$;
- $R = \{1, 21, 321, \ldots\}$;
- if $X$ and $Y$ are closed then so is $X \cup Y$. 

Defining Pattern Classes

- basis;
- token passing networks (TPNs);
- constructions:
  - union;
  - direct sum;
  - juxtaposition;
  - merge;
  - composition;
  - ...
- Sub() operator.
Properties

Definition/representation:

- finitely based;
- generated by a TPN;
- defined by Sub().

Enumeration:

- (a nice) recurrence;
- rational/algebraic/D-finite generating function;
- growth rate.

Structure:

- finite
- sum complete;
- atomic (union indecomposable);
- partially well ordered;
- . . .
Basis

**Definition.** The basis $\mathcal{B}(X)$ of a closed class $X$ is the set of minimal elements not in $X$:

$$\mathcal{B}(X) = \{ \sigma \notin X : \forall \tau : \tau \prec \sigma \Rightarrow \tau \in X \}.$$ 

**Example.** $\mathcal{B}(S) = \emptyset$; $\mathcal{B}(I) = \{21\}$; $\mathcal{B}(I \cup R) = \{132, 213, 231, 312\}$.

**Definition.** For a set $Z \subseteq S$ its avoidance set is

$$\mathcal{A}(Z) = \{ \sigma : \forall \tau \in Z : \tau \nleq \sigma \}.$$ 

**Facts.**

- $\mathcal{B}(X)$ is an antichain.
- $\mathcal{A}(Z)$ is closed.
- $\mathcal{A}(\mathcal{B}(X)) = X$. 
Basis $\rightarrow$ Finite

**Theorem.** A closed class $X$ is finite iff its basis contains $12\ldots m$ and $n\ldots 21$ for some $m$ and $n$. (And hence finiteness is decidable from the basis.)

**Proof.** ($\Leftarrow$) Erdös, Szekeres.

($\Rightarrow$) $X$ finite $\Rightarrow \exists m : 12\ldots m \notin X$. Choose the smallest such $m$. Then $12\ldots m \in \mathcal{B}(X)$. 
Direct Sum

**Definition.** For $\sigma = s_1 \ldots s_m$ and $\tau = t_1 \ldots t_n$ define

$$\sigma \oplus \tau = s_1, \ldots, s_m, t_1 + m, \ldots, t_n + m.$$ 

**Example.** $132 \oplus 213 = 132546$.

**Definition.** $X \oplus Y = \{\sigma \oplus \tau : \sigma \in X, \, \tau \in Y\}$.

**Fact.** $X, Y$ closed $\Rightarrow X \oplus Y$ closed.

**Examples.** $I \oplus I = I$; $R \oplus R = A(123, 312, 231)$.

**Definition.** $X$ is sum complete if $\sigma, \tau \in X \Rightarrow \sigma \oplus \tau \in X$.

**Examples.** $I$ is sum complete, $R$ is not.
Definition. \( \sigma \) is sum indecomposable if \( \sigma \neq \tau \oplus \pi \) for any (non-empty) \( \tau, \pi \).

Theorem. A closed class \( X \) is sum complete iff every element of its basis is sum indecomposable.

Proof. (\( \Rightarrow \)) Suppose \( \sigma = \tau \oplus \pi \in B(X) \). Then \( \tau, \pi \prec \sigma \) implies \( \tau, \pi \in X \), but \( \tau \oplus \pi \notin X \).

(\( \Leftarrow \))

\[
\begin{align*}
\sigma, \tau & \in X \\
\iff & \forall \beta \in B(X) : \beta \not\preceq \sigma \& \beta \not\preceq \tau \\
\iff & \forall \beta \in B(X) : \beta \not\preceq \sigma \oplus \tau \\
\iff & \sigma \oplus \tau \in X.
\end{align*}
\]
Sub() Operator

**Definition.** Let $A, B$ be (usually infinite) linearly ordered sets, and let $\pi : A \to B$ be a bijection. For every finite $C \subseteq A$, $\pi|_C$ is order isomorphic to a permutation. The set of all such permutations is denoted by $\text{Sub}(\pi : A \to B)$ or just $\text{Sub}(\pi)$.

**Fact.** $\text{Sub}(\pi)$ is closed.

**Example.** $\pi : \mathbb{N} \to -\mathbb{N}$, $\pi(x) = -x$. $\text{Sub}(\pi) = \mathbb{R}$.

**Example.** $\pi : \mathbb{N} \to \mathbb{N}$, $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$, $4 \mapsto 4$, ... $\text{Sub}(\pi) = \{12...n : n \geq 1\} \cup \{2134...n : n \geq 2\}$. 
Unions, Sub and Join Property

**Theorem.** [Atkinson, Murphy, NR] The following conditions are equivalent for a closed class $X$:

(i) $X \neq Y \cup Z$ for any proper subclasses $Y, Z$.

(ii) $X = \text{Sub}(\pi : A \to B)$ for some $\pi, A, B$.

(iii) $X$ has the join property:

$$\forall \sigma, \tau \in X : \exists \pi \in X : \sigma \preceq \pi \ & \tau \preceq \pi.$$ 

**Definition.** A class satisfying one (and hence all) of the above conditions is said to be atomic.
Unions, Sub and Join Property

**Example.** $I$, $R$, $\text{Sub}(2134\ldots)$ and $R \oplus R$ are atomic. $I \cup R$ is not atomic.

**Fact.** Sum complete $\Rightarrow$ atomic.

**Open Question.** Is atomicity decidable from the basis?
Sub($\pi : \mathbb{N} \rightarrow \mathbb{N}$) (Natural Classes)

**Fact.** Sum complete $\Rightarrow$ natural.

**Fact.** $\gamma \in S$, $X$ sum complete $\Rightarrow$ $\text{Sub}(\gamma) \oplus X$ natural.

**Theorem.** [Atkinson, Murphy, NR] If $X$ is a finitely based natural class then one of the following holds:

(i) $X = \text{Sub}(\gamma) \oplus Y$, where $\gamma \in S$, $Y$ is sum complete and uniquely determined by $X$; or

(ii) $X = \text{Sub}(\pi : \mathbb{N} \rightarrow \mathbb{N})$ where $\pi$ is uniquely determined by $X$ and is ultimately periodic:

$$\exists N, \omega : \forall n \geq N : \pi(n + \omega) = \pi(n) + \omega.$$


Basis $\rightarrow$ Natural

**Corollary.** [Murphy] It is decidable whether a finitely based closed class is natural.

**Open Questions.** Is it decidable whether a finitely based class can be written as any of the following:

- $\text{Sub}(\pi : \mathbb{Z} \rightarrow \mathbb{Z})$?
- $\text{Sub}(\pi : 2\mathbb{N} \rightarrow \mathbb{N})$?
- $\text{Sub}(\pi : \mathbb{Q} \rightarrow \mathbb{Q})$?
Enumeration

\[ s_n(X) = \text{the number of permutations of length } n \text{ in } X. \]

**Example.** \[ s_n(I) = s_n(R) = 1; \ s_n(R \oplus R) = n. \]

**Theorem.** [Knuth; Simion, Schmidt] If \( \pi \) is a permutation of length 3 then \( s_n(A(\pi)) = C_n \), the \( n \)th Catalan number.

**Remarks.** Exact formulae are known for \( s_n(A(\pi)) \) where \( \pi \) is any permutation of length 4, except when \( \pi \) is (equivalent to) 1324. Very little is known about \( s_n(A(\pi)) \) when \( |\pi| \geq 5 \).
Enumeration: Generating Functions

Open Questions. Is it decidable whether the ordinary generating function of $s_n(X)$ for a finitely based closed class $X$ is (a) rational? (b) algebraic? (c) $D$-finite?

Theorem. [Bousquet-Melou] The generating function for $A(1234)$ is $D$-finite but not algebraic.

Theorem. [Gessel] Is the generating function for a finitely based closed class always $D$-finite?

Theorem. [Murphy] There exists an infinitely based closed class the generating function of which is not $D$-finite.
Enumeration: Growth

**Theorem.** If $X \neq S$ is a closed class then there exists $q$ such that $s_n(X) \leq q^n$.

**Remarks.** Proved very recently by Marcus and Tardos. Before that it was known as the Wilf–Stanley Conjecture.

**Corollary.** The limit $q = \lim_{n \to \infty} \sqrt[n]{s_n(X)}$ exists. (The growth of $X$.)

**Examples.** The growth of $\mathcal{A}(\pi)$ where $|\pi| = 3$ is 4. [Regev] The growth of $\mathcal{A}(12\ldots k)$ is $(k - 1)^2$.

**Questions.** For a fixed $q$, is it decidable whether $q$ is the growth of a finitely based class $X$? Is the growth of a finitely based class effectively computable?

**Remark.** [Bona] The growth of $\mathcal{A}(12453)$ is $9 + 4\sqrt{2}$. 
**Token Passing Networks (TPNs)**

A TPN is a finite directed graph with a distinguished input vertex I and a distinguished output vertex O. Each vertex is one of the following:

- simple node, capable of holding one item of data;
- stack, capable of holding any number of items, and treating them in the FILO discipline;
- queue (FIFO);
- ...  

If all vertices are of type (i) we have a finite capacity (FC) TPN.
Let $N$ be a TPN. $N$ can generate permutations: feed $12\ldots n$ into $N$, item by item, through $I$, move items along edges respecting the orientation, store them in vertices respecting the type, output them via $O$.

$\mathcal{P}(N) = \text{the set of all output permutations.}$

**Facts.** $\mathcal{P}(N)$ is closed, sum complete and atomic.

**Remark.** Now atomicity is decidable!
$X = \mathcal{P}(N)$

- $X$ is not finitely based.
- Unknown: $\mathcal{B}(X)$; $s_n(X)$; growth; enumeration for the basis.
FCTPNs

Let $N$ be a FCTPN, $|N| = m$, $X = \mathcal{P}(N)$.

**Theorems.** (i) [Atkinson, Livsey, Tulley] Elements of $N$ can be encoded, symbol by symbol, by words over an alphabet of size at most $m$. [The actual size of the alphabet will be called the boundedness of $X$.] The resulting set of words is a regular language.

(ii) The generating function for $X$ is rational and can be effectively computed.

(iii) [Albert, Atkinson, NR] $\mathcal{B}(X)$ can be encoded in the same fashion, over the same alphabet, again yielding a regular language.

(iv) It is decidable whether $X$ is finitely based.

**Remark.** The above algorithms are practical and have been implemented in GAP.
**FCTPNs**

**Theorem.** [Albert, Linton, NR] For any $m$ there are only finitely many pattern classes of the form $\mathcal{P}(N)$, where $N$ is a FCTPN of boundedness $m$.

**Fact.** If a class $X$ is given by its basis, it is decidable whether $X$ is bounded.

**Question.** Is it decidable whether a given finitely based class is generated by a FCTPN?
TPNs

**Question.** Is it decidable whether a given TPN generates a finitely based class?

**Theorem.** [Waton] It is decidable whether a given TPN generates $S$, the class of all permutations.

**Idea.** Avoid
Partial Well Order:
Towards Undecidability?

Definition. [Following G. Higman] A pattern class $X$ is said to be partially well ordered (PWO) if it contains no infinite antichain.

Theorem. [Atkinson, Murphy, NR] A finitely based closed class is PWO iff it has only countably many subclasses.

Examples.

- $\mathcal{A}(12)$ is PWO.
- [Atkinson, Murphy, NR] $\mathcal{A}(231)$ is PWO.
- [Spielman, Bona] $\mathcal{A}(123)$ is not PWO.

Questions. Is partial well orderedness decidable for closed classes given by (a) finite bases? (b) TPNs? (c) FCTPNs?
> f := proc(list)
> RETURN(true);
> end;

> f := proc(list) RETURN(true) end proc

> f([2,3,1],[3,5,2,1,4]);
  true