PRESENTATIONS OF (SUB)SEMIGROUPS

or

Presentations for semigroups and their subsemigroups

or

Relationships between presentations (generators and relations) for a semigroup and for its subsemigroups

or

What are relationships between presentations defining a semigroup and those defining its subsemigroups?

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Surely,

if S is finitely presented

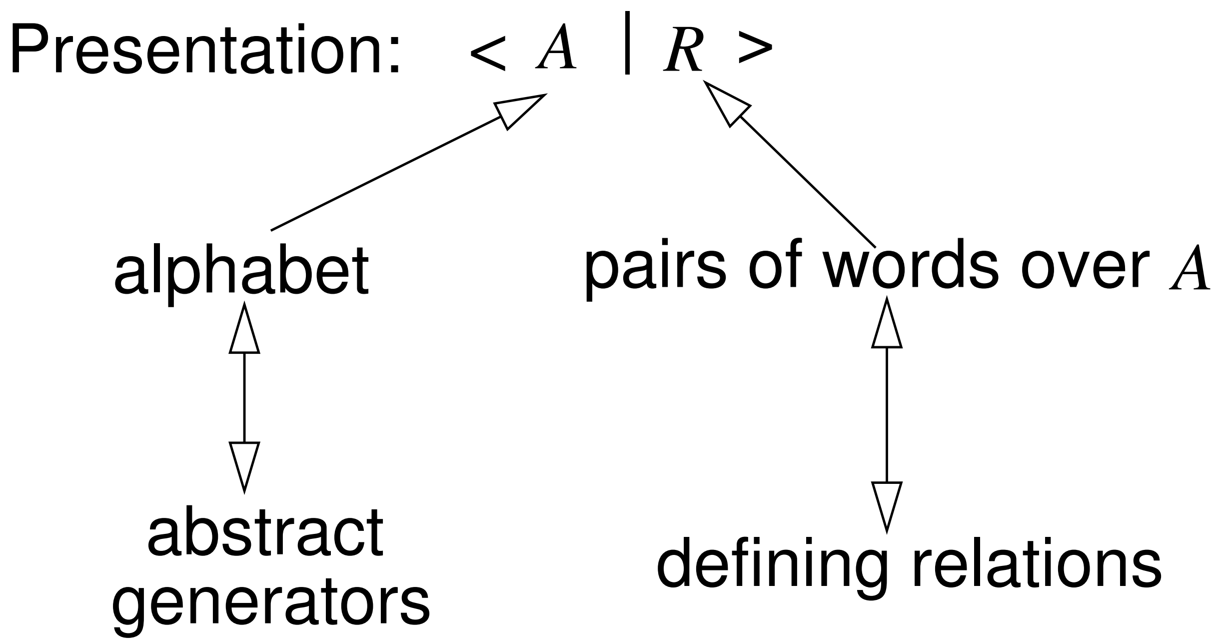
$(S = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ldots, u_n = v_n \rangle)$

and if T is finitely generated (by $b_1, \ldots, b_p$)

then T must be finitely presented

(because we just need to translate the relations $u_1 = v_1$, ..., $u_n = v_n$ into the alphabet $b_1, \ldots, b_p$)

?



Defines the semigroup $S \cong A^+/\rho$, where ρ is the smallest congruence on A^+ containing R.

S is the largest (in terms of homomorphic images) semigroup generated by A in which the generators satisfy all relations from R.

S is finitely generated if A can be chosen to be finite.

S is finitely presented if both A and R can be chosen to be finite.

Example. $\langle A \mid \ \rangle$ defines the free semigroup A^+.

Example. $\langle a \mid a^2 = a \rangle$ defines the trivial semigroup.

Example. $\langle a, b \mid ab = ba \rangle$ defines the free commutative semigroup.

Example. $\langle a, a', b, b' \mid aa' = a'a = bb' = b'b = 1 \rangle$ (monoid presentation!) defines the free group on $\{a, b\}$.

Example. $\langle b, c \mid bc = 1 \rangle$ defines the bicyclic monoid, in which every element can be written uniquely as $c^i b^j$ $(i, j \geq 0)$.

Example. $S = \langle a, b \mid ab^i a = aba \ (i = 2, 3, \ldots) \rangle$ is not finitely presented.

Example

Let $S = \{a, b, c\}^+$, and let T be the subsemigroup of S generated by

$$x = ab, \qquad y = b^2c,$$
$$m = b^3,$$
$$z = ab^2, \qquad t = bc.$$

Then T is defined by

$$\langle x, y, z, t, m \mid xm^iy = zm^it \ (i = 0, 1, 2, \ldots)\rangle.$$

T is not finitely presented.

So, a finitely generated subsemigroup of a finitely presented (even free) semigroup is not necessarily finitely presented.

Higman's Embedding Theorem(s)

Theorem. [Higman (1961)] A finitely generated group H is a subgroup of a finitely presented group if and only if H is recursively presented (i.e. $\langle A \mid R \rangle$, R recursively enumerable).

Theorem. [Murski, 1967] The same as above, just replace group $\longrightarrow$ semigroup.

Our question is becoming:

When (under which interesting conditions) is a finitely generated subsemigroup of a finitely generated semigroup itself finitely presented?

Subgroups (of groups)

Theorem. [Reidemeister–Schreier] Let G be a group, and let H be a subgroup of finite index in G.

(i) G finitely generated $\iff$ H finitely generated;

(ii) G finitely presented $\iff$ H finitely presented.

Notes about the proof.

- ($\Rightarrow$) is the harder part.
- The proof of f.g. result not only gives a generating set B for H in terms of a generating set A for G, but also a 'nice' mapping which rewrites words over A representing elements of T into corresponding words over B.
- The f.p. part is then proved by rewriting defining relations for G into the new alphabet B.

Subgroups (of semigroups)

Definition. Let S be a semigroup, and let H be a subgroup of S. A coset of H is a set of the form Hs ($s \in S$) such that $Hst = H$ for some $t \in S$.

Theorem. [NR, 1999] Let S be a semigroup, and let H be a subgroup of S with finitely many cosets.

(i) S finitely generated $\Rightarrow$ H finitely generated;

(ii) S finitely presented $\Rightarrow$ H finitely presented.

Notes.

- The proof is basically the same as the proof of Reidemeister–Schreier.
- The converse implications, of course, do not hold.

Group of units

Definition. If S is a monoid, the group of units $U(S)$ of S consists of all (left and right) invertible elements of S.

Theorem. [Zhang, 1992] Let S be a monoid defined by a presentation of the form

$$\langle a_1, \ldots, a_m \mid u_1 = 1, \ldots, u_n = 1 \rangle.$$

Then the group of units $U(S)$ is finitely presented.

Fact. If S is a finitely presented monoid in which all left invertible elements are also right invertible then its group of units is also finitely presented.

Group of units: example

Let S be defined by

$$\langle a_i, a_i' \ (i = 1, 2, 3, 4), b, c \mid$$
$$a_i a_i' = a_i' a_i = 1, \ a_1 a_2 = a_3 a_4, \ bc = 1,$$
$$b a_i = a_i^2 b, \ a_i c = c a_i^2 \ (i = 1, 2, 3, 4)\rangle.$$

The group of units $U(S)$ is defined by

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^j} a_2^{2^j} = a_3^{2^j} a_4^{2^j} \ (j = 0, 1, 2, \ldots)\rangle$$

(group presentation) and is not finitely presented.

Question. Does there exist a finitely presented monoid, the group of units of which is not finitely generated?

Finite complement (generators)

Theorem. [Jura, 1978] Let S be a semigroup, and let T be a subsemigroup of S with $S \setminus T$ finite. Then T is finitely generated iff S is finitely generated.

Proof. ($\Rightarrow$) $T = \langle X \rangle \Rightarrow S = \langle X \cup (S \setminus T) \rangle$.

($\Leftarrow$) Suppose $S = \langle Y \rangle$. Let $t \in T$ be arbitrary and write $t = y_1 y_2 \ldots y_n$ ($y_i \in Y$). Let i be the smallest such that $y_1 \ldots y_i \in T$. Consider $t' = y_{i+1} \ldots y_n$. If $t' \in T$ then repeat the above; otherwise stop. Hence

$$T = \langle \{uyv \ : \ u, v \in (S \setminus T)^1,\ y \in Y,\ uy \in T,\ uyv \in T\} \rangle.$$

Finite complement (presentations)

Theorem. [NR, 1998] Let S be a semigroup, and let T be a subsemigroup of S with $S \setminus T$ finite. Then T is finitely presented iff S is finitely presented.

Notes about the proof.

- (Only) formally similar to Reidemeister–Schreier.
- Surprisingly, technically (much) more complicated than R–S.
- Based on a rewriting mapping arising from the proof of finite generation theorem.

Inverse semigroups

Definition. A semigroup S is inverse if for every $s \in S$ there exists a unique $t(= s^{-1})$ such that $sts = s$, $tst = t$.

Fact. Inverse semigroups form a variety (with signature $(\cdot, {}^{-1})$). Hence there are free inverse semigroups (lovely objects) and one can talk about inverse semigroup presentations.

A word of caution. [Schein, 1975] Free inverse semigroups (even the monogenic ones) are not finitely presented as semigroups (but, of course, are as inverse semigroups).

Inverse semigroups: finite complement

> **Theorem** [Araujo, NR, Silva] Let S be an inverse semigroup, and let T be an inverse subsemigroup of S with $S \setminus T$ finite. Then T is finitely presented iff S is finitely presented (both as inverse semigroups).

Notes about the proof.

- Not an immediate consequence of the (ordinary) semigroup result.
- Again, surprisingly difficult, even modulo the semigroup result.
- Uses the semigroup result in a somewhat surprising (syntactic, rather than semantic) way.

A decidability question

Question. Is there an algorithm which does the following:

INPUT:

- A finite presentation $\langle A \,|\, \rangle$ (defining S);
- a finite set $X \subseteq A^+$ (generating a sub-semigroup T of S).

OUTPUT:

- YES if T is finitely presented;
- NO otherwise?

Answer: NO (it does not exist).

But

What if we fix $\langle A | R \rangle$ (to be something nice and simple) in advance?

Subsemigroups of free semigroups

Theorems [Sardinas, Patterson 1953; Spehner 1974; et al.] Algorithms for deciding whether a finitely generated subsemigroup of a free semigroup is free.

Theorem [Markov 1971] It is decidable whether a finitely generated subsemigroup of a free semigroup is finitely presented.

Notes about the proof.

- From the given generating set (for the subsemigroup T) compute a language which represents 'minimal' relations holding in T.
- This language turns out to be regular, and so it is decidable whether or not it is finite.

Subsemigroups of free groups

Theorem [Nielsen–Schreier] Every subgroup of a free group is free.

Theorem. [Cain, Robertson, NR] It is decidable whether a finitely generated subsemigroup of a free group is free.

Note about the proof. Here context-free languages are employed.

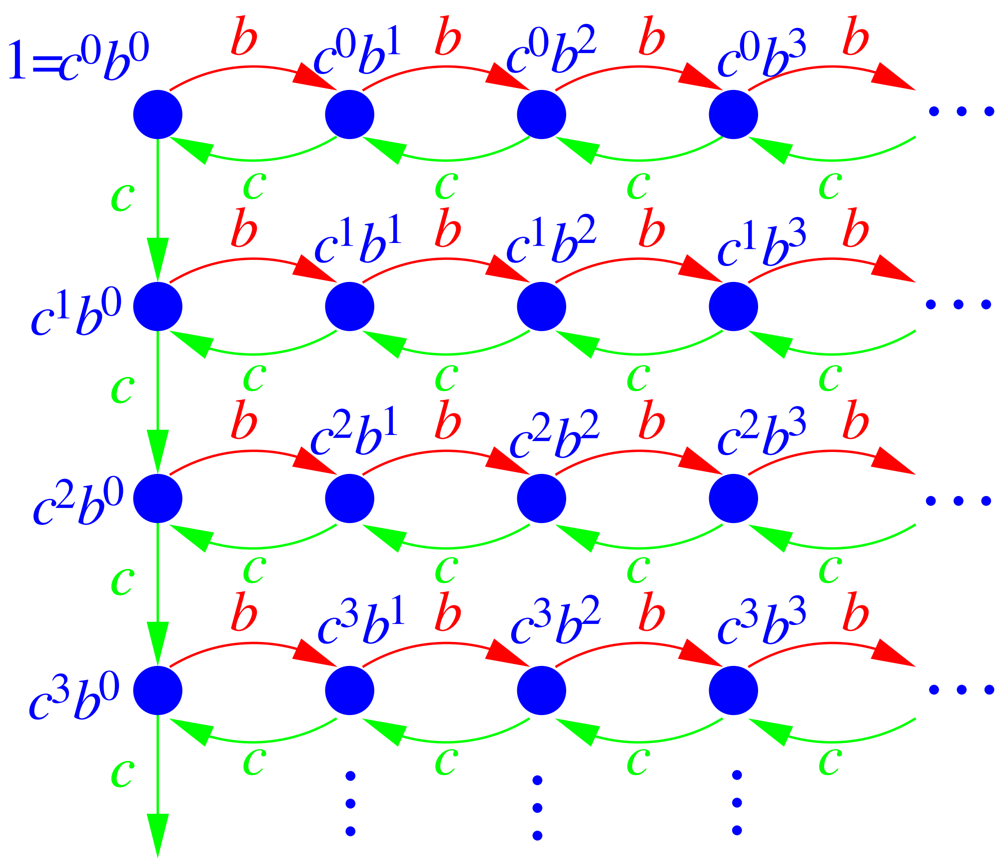
Question. Is it decidable whether a finitely generated subsemigroup of a free group is finitely presented?

Cayley graphs

Setting: $S = \langle A \rangle$, A finite; $T \leq S$.

Definition. The (right) Cayley graph of S with respect to A: $\Gamma(S, A) = (V, E)$, $V = S$, $E = \{s \xrightarrow{a} sa \;:\; s \in S,\; a \in A\}$. The left Cayley graph is defined dually.

Example. Bicyclic monoid:



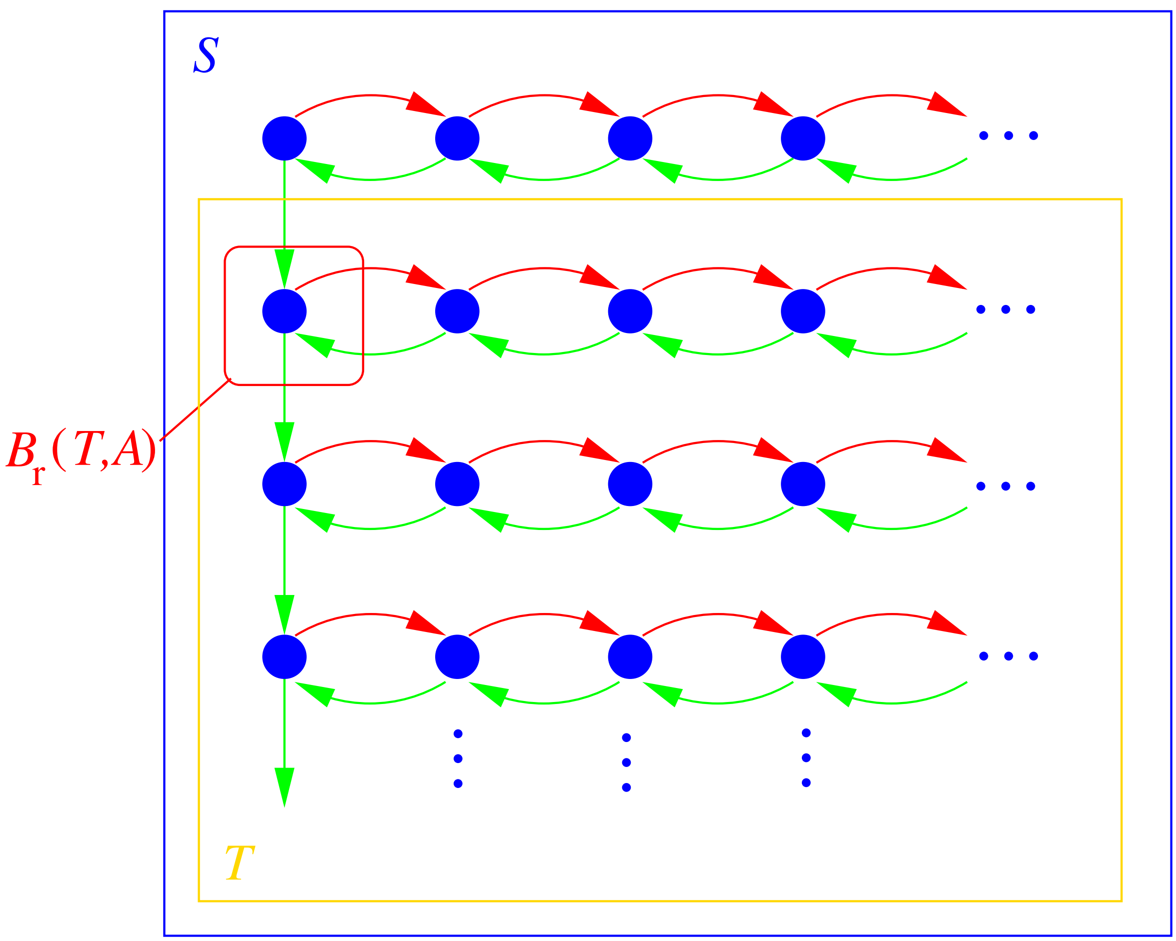
Boundaries (Gray, NR)

Definition. The right boundary of T (in S w.r.t. A) is the set of all elements of T which, in the right Cayley graph, receive an arrow from outside:

$$B_r(T,A) = T \cap (S \setminus T)A.$$

Likewise, the left boundary:

$$B_l(T,A) = T \cap A(S \setminus T).$$



Boundaries

Example. If $S = \{a, b\}^+$, $A = \{a, b\}$, $T = \{w : |w| \geq 3\}$ then

$$B_r(T, A) = B_l(T, A) = \{w : |w| = 3\}.$$

Fact. Each of these is sufficient (but not necessary) for $B_r(T, A)$ and $B_l(T, A)$ to be finite:

- T finite;
- $S \setminus T$ finite (finite complement);
- $S \setminus T$ an ideal.

Boundaries: basic properties

Theorem. (Invariance) Finiteness of a boundary depends on S and T, and not on the chosen generating set A.

Theorem. Let S be a finitely generated semigroup, and let T be a subsemigroup of S. Then T has finite boundaries in S iff:

(i) for every finite $X \subseteq S$ the sets $(S \setminus T)^1 X \cap T$ and $X(S \setminus T)^1 \cap T$ are finite; and

(ii) $(S \setminus T)^2 \cap T$ is finite.

Fact. It is not true that if $U \leq T \leq S$, with T having finite boundaries in S, and U having finite boundaries in T, then U must have finite boundaries in S. One example is provided by $S = \langle a, b, c \mid ba = 0,\ ca = 0,\ cb = 0 \rangle$, $T = \langle a, b, ab \rangle$, $U = \langle a, abc \rangle$.

Boundaries: generators

Theorem. If $T \leq S = \langle A \rangle$, A finite, then

$$T = \langle B_r(A,T)(S \setminus T) \cap T \rangle.$$

Note on proof. Basically the same as Jura's proof in the finite complement (generators) case.

Corollary. If T has finite boundaries in (a finitely generated semigroup) S then T is finitely generated.

Boundaries: presentations

Theorem. If T has finite boundaries in a finitely presented semigroup S then T is finitely presented.

Notes on proof.

- Along similar lines as finite complement.
- Only a little bit more complicated, but also a bit less obscure.
- Corrects a mistake in the original finite complement proof.

Further progress: more questions

Question. Let T be a completely regular (=union of groups) subsemigroup of a completely regular semigroup S such that $S \setminus T$ is finite. Is it true that T is finitely generated (respectively finitely presented) iff S is finitely generated (resp. finitely presented)?

Remark. Both finite generation and presentability above are to be understood in the completely regular sense. Both are different from the corresponding notions in the semigroup sense.

Further progress: more questions

Question. Is every finitely generated inverse subsemigroup of a free inverse semigroup finitely presented (as an inverse semigroup)? If not, is there an algorithm which decides finite presentability?

Meta-problem. Formulate and prove a theorem which would have as its special cases as many as possible of the following:

- Reidemeister–Schreier for groups;
- Reidemeister–Schreier for semigroups;
- finite complement (f.p.);
- finite boundaries (f.p.).

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