

Residual Finiteness of Monoids, Associated Actions and Groups

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SAGE, University of Essex, 28 Aug 2008



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Residual Finiteness (1)

Definition

An algebraic structure A is **residually finite** if any, and hence all, of the following equivalent conditions hold:

- ▶ For all $x, y \in A$, $x \neq y$, there exists a homomorphism $f : A \rightarrow B$, B finite, such that $f(x) \neq f(y)$.
- ▶ For all $x, y \in A$, $x \neq y$, there exists a finite index congruence ρ which separates x and y , i.e. $(x, y) \notin \rho$.
- ▶ The intersection of all finite index congruences of A is trivial.

This applies to: groups, semigroups/monoids, actions, ...



Residual Finiteness (2)

Examples

- ▶ finite structures (r.f. is a finiteness condition);
- ▶ free – semigroups/ groups/ commutative semigroups/ abelian groups/inverse semigroups;
- ▶ infinite simple (congruence free) structures are not r.f.

Facts

- ▶ *Closed under taking substructures (obvious).*
- ▶ *Closed under finite index extensions (in groups), finite Rees index extensions (in semigroups, NR, Thomas), direct products (in general).*
- ▶ *A finitely presented, r.f. algebraic structure has a soluble word problem (Mostowski 66, Evans 70).*



Residual Finiteness (3)

- ▶ W Magnus, Residually Finite Groups, 1969.
- ▶ D. Segal, Residually finite groups, 1990.
- ▶ A series of papers by Golubov et al., 1970s.



Right Regular Representation and Green's Equivalences \mathcal{R} , \mathcal{L} , \mathcal{H}

Any monoid S acts on itself via $(x, s) \mapsto xs$.

\mathcal{R} -classes := the strong orbits of this action.

\mathcal{R} is a left congruence (i.e. $s \in S$ & $(x, y) \in \mathcal{R} \Rightarrow (sx, sy) \in \mathcal{R}$).

Hence, S acts from the left on the set S/\mathcal{R} of \mathcal{R} -classes.

Left/right duality \rightarrow \mathcal{L} -classes, right action on S/\mathcal{L} .

$\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.



Schützenberger Groups (1)

Let $h \in S$, and let H be the \mathcal{H} -class of h .

$$\text{Stab}_r(h) = \{s \in S : Hs = H\} \leq S.$$

$s \sim t \Leftrightarrow hs = ht$ – a congruence on $\text{Stab}_r(h)$.

$$\Gamma_r(h) = \text{Stab}_r(h) / \sim.$$

$\Gamma_l(h)$ – defined dually.



Schützenberger Groups (2)

Facts

- ▶ $\Gamma_r(h)$ is a group acting regularly on H .
- ▶ $|\Gamma_r(h)| = |H|$.
- ▶ $\Gamma_r(h) \cong \Gamma_l(h)$.
- ▶ $(h, h') \in \mathcal{R} \Rightarrow \Gamma_r(h) \cong \Gamma_l(h')$.



Idempotents, Subgroups, Regular Monoids

Fact

If $h \in S$ is an idempotent then its \mathcal{H} -class H is the largest subgroup of S containing h and $\Gamma_r(h) \cong H$.

Definition

A monoid S is **regular** if $(\forall x)(\exists y)(xyx = x)$.

Fact

A monoid is regular iff every \mathcal{R} -class contains an idempotent.



In a Nutshell ...

A monoid is 'composed' out of two actions (on S/\mathcal{R} and S/\mathcal{L}) and a bunch of groups ($\Gamma_r(h)$, $h \in H$).

Is the following in any sense true:

S is residually finite if and only if the actions on S/\mathcal{R} and S/\mathcal{L} , and all the Schützenberger groups $\Gamma_r(h)$ ($h \in H$) are residually finite?

😊 After all: S is finite iff S/\mathcal{R} , S/\mathcal{L} and all $\Gamma_r(h)$ are finite and r.f. is a finiteness condition. 😊

Positive Results: Regular Monoids (1)

Definition

A monoid S is of **finite \mathcal{J} -type** if every \mathcal{J} -class contains only finitely many \mathcal{R} - and \mathcal{L} -classes.

Theorem (Golubov 75)

Let S be a regular monoid of finite \mathcal{J} -type. Then S is residually finite if and only if all its maximal subgroups (i.e. Schützenberger groups) are residually finite.

Proposition

If a regular monoid S is of finite \mathcal{J} -type then its action on its \mathcal{R} -classes is also residually finite.



Positive Results: Regular Monoids (2)

Theorem

The action of a residually finite regular monoid on its \mathcal{L} -classes is residually finite.

Sketch of Proof

Let L_s, L_t be two distinct \mathcal{L} -classes.

Let e, f be idempotents s.t. $e \in L_s, f \in L_t$.

General theory: $(e, f) \notin \mathcal{L} \Leftrightarrow ef \neq e \vee fe \neq f$.

Wlog suppose $ef \neq e$.

Separate e, ef : $\phi : S \rightarrow T, T$ finite, $\phi(ef) \neq \phi(e)$.

Hence, in $T, (\phi(e), \phi(f)) \notin \mathcal{L}$.

The action of S on the \mathcal{L} -classes of T is a finite homomorphic image of the action of S on S/\mathcal{L} which separates L_s and L_t .



Positive Results: General Monoids

Theorem

If S is a residually finite monoid then every Schützenberger group of S is residually finite.

Sketch of Proof

(sketch) Let \bar{s}, \bar{t} be distinct elements of $\Gamma_r(h)$.

That means that $hs \neq ht$ in S .

Separate hs, ht : $\phi : S \rightarrow T$, T finite, $\phi(hs) \neq \phi(ht)$.

This induces $\Gamma_r(h) \rightarrow \Gamma_r(\phi(h))$ which separates \bar{s} and \bar{t} .



Summary (1)

Properties of M		M r.f. \Rightarrow all $\Gamma(H)$ r.f.	M r.f. \Rightarrow M/\mathcal{L} r.f.	all $\Gamma(H)$ r.f. & $M/\mathcal{L}, M/\mathcal{R}$ r.f. $\Rightarrow M$ r.f.
general	arbitrary	✓		
	regular	✓	✓	
finite \mathcal{J} -type	arbitrary	✓		
	regular	✓	✓	✓

Rees Matrix Semigroups (1)

Ingredients:

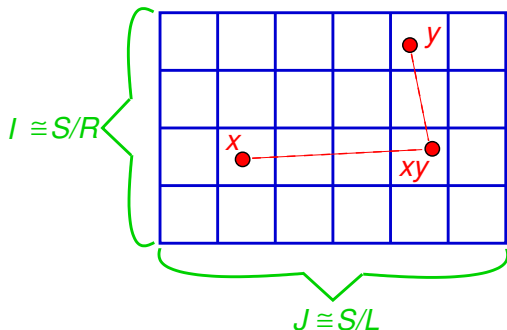
- ▶ a group G ;
- ▶ two index sets I, J ;
- ▶ $P = (p_{ji})_{j \in J, i \in I}$ – a $J \times I$ matrix with entries from G ;

New semigroup: $S = M[G; I, J; P]$, on the set $I \times G \times J$, with multiplication

$$(i, g, j)(k, h, l) = (i, gp_{jk}h, l).$$



Rees Matrix Semigroups (2)



Facts

- ▶ S is regular.
- ▶ \mathcal{L} -classes are indexed by the set J .
- ▶ Every element acts as a constant mapping on S/\mathcal{L} .
- ▶ The action of S on S/\mathcal{L} is always residually finite.

Rees Matrix Semigroups (3)

Theorem (Golubov 72)

A Rees matrix semigroup $M[G; I, J; P]$ is residually finite if and only if G is residually finite and P has only finitely many non-proportional rows and columns.

Corollary

There exists a non-residually finite regular semigroup in which the actions on S/\mathcal{R} and S/\mathcal{L} , and all the maximal subgroups are residually finite.



Another Construction (1)

Ingredients: A group G and a normal subgroup $N \trianglelefteq G$.

Let $\bar{N} = \{\bar{n} : n \in N\}$ be a copy of N .

New semigroup:

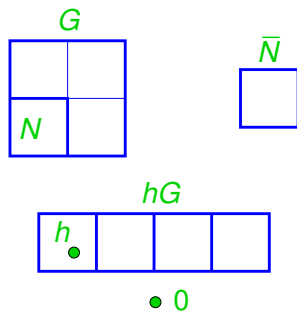
$$\mathcal{S}(G, N) = \langle G, \bar{N}, h : hn = \bar{n}h, he_G = e_{\bar{N}}h = h, \\ g\bar{n} = \bar{n}g = gh = h\bar{n} = 0 (g \in G, n \in N) \rangle.$$



Another Construction (2)

Facts

- ▶ $S(G, N)$ is r.f. iff G is r.f.
- ▶ The action of $S(G, N)$ on its \mathcal{L} -classes is r.f. iff G/N is r.f.



Corollary

There exists a residually finite semigroup such that its action on the \mathcal{L} -classes is not residually finite.

One More Counter-example (1)

Let M be the commutative monoid with presentation

$$\begin{aligned} & \langle a, a^{-1}, b_i, c_i, d, e \ (i \in \mathbb{Z}) \mid \\ & aa^{-1} = a^{-1}a = 1, \ b_i c_i = d, \ b_i c_j = a^{\tau(j-i)} e \ (i \neq j), \\ & b_i b_j = b_i c_j = b_i d = b_i e = c_j c_k = c_j d = c_j e = dd = de = ee = 0 \\ & \quad (i, j, k \in \mathbb{Z}) \rangle \end{aligned}$$

The \mathcal{R} -/ \mathcal{L} -classes of M are:

$$A = \{a^{\pm p} : p \in \mathbb{Z}\} - \text{the group of units}$$

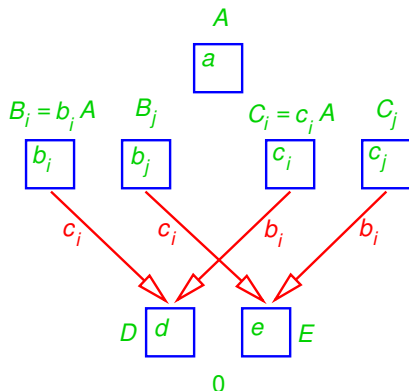
$$B = \cup_{i \in \mathbb{Z}} B_i, \ B_i = Ab_i$$

$$C = \cup_{i \in \mathbb{Z}} C_i, \ C_i = Ac_i$$

$$D = Ad, \ E = Ae, \ \{0\}.$$



One More Counter-example (2)



The action of M on M/\mathcal{L} is not residually finite.

One More Counter-example (3)

We want M to be residually finite.

We still have one free choice: the function $\tau : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Z}$.

One choice that works is:

$$\tau(2^k(2r + 1)) = \frac{2}{3}(2^{2^{\lceil k/2 \rceil}} - 1) \quad (k, r \in \mathbb{Z}, k \geq 0) \quad \text{😊}$$



Finitely Many Ideals

Let's introduce a really strong finiteness condition on the actions on S/\mathcal{R} and S/\mathcal{L} : finiteness itself.

😊 That certainly guarantees that the two actions will be residually finite. 😊

Theorem

Let S be a monoid with finitely many left- and right ideals. Then S is residually finite if and only if all its Schützenberger groups are residually finite.



Summary (2)

Properties of M		M r.f. \Rightarrow all $\Gamma(H)$ r.f.	M r.f. \Rightarrow M/\mathcal{L} r.f.	all $\Gamma(H)$ r.f. & $M/\mathcal{L}, M/\mathcal{R}$ r.f. $\Rightarrow M$ r.f.
general	arbitrary	✓	✗	✗
	regular	✓	✓	✗
finite \mathcal{J} -type	arbitrary	✓	✗	✗
	regular	✓	✓	✓
$< \infty$ many left/right ideals	arbitrary	✓	✓	✓
	regular	✓	✓	✓