

# **Decidability Questions for Pattern Avoidance Classes of Permutations**

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## Template Question

Does there exist an algorithm which does the following:

**INPUT:** A (finite)

(def)

---

defining a pattern avoidance class  $X$  of permutations.

**OUTPUT:** TRUE if  $X$  is

(property)

---

FALSE otherwise?

If such an algorithm exists we say that the property is decidable; if not, it is undecidable.

## Defining Pattern Classes

- basis;
- token passing networks (TPNs);
- Sub() operator;
- constructions:
  - union;
  - direct sum;
  - juxtaposition;
  - merge;
  - composition;
  - ...

# Properties

Definition/representation:

- finitely based;
- generated by a TPN;
- defined by  $\text{Sub}()$ .

Encoding:

- regular/context sensitive;
- finitely labelled generating tree.

Enumeration:

- (a nice) recurrence;
- rational/algebraic/D-finite generating function;
- growth rate.

Structure:

- finite;
- atomic (=union indecomposable);
- sum complete;
- ...

## Basis $\rightarrow$ Finite

**Theorem.** A closed class  $X$  is finite iff its basis contains  $12\dots m$  and  $n\dots 21$  for some  $m$  and  $n$ . (Finiteness is decidable from the basis.)

**Proof.** ( $\Leftarrow$ ) Erdős, Szekeres.

( $\Rightarrow$ )  $X$  finite  $\Rightarrow \exists m : 12\dots m \notin X$ . Choose the smallest such  $m$ . Then  $12\dots m \in \mathcal{B}(X)$ .

## Basis $\rightarrow$ Atomic?

Max says: To be atomic is to have a strong and intuitive property.

Definition. A pattern class  $X$  is said to be atomic if it is not a union of two proper subclasses:  $X = Y \cup Z \Rightarrow X = Y$  or  $X = Z$ .

Open Problem. Is it decidable whether a given finitely based class is atomic?

## Atomicity

**Fact.**  $\mathcal{A}(\alpha)$  is atomic for every  $\alpha$ .

**Example.**  $\mathcal{A}(132, 4321) = \mathcal{A}(132, 4321, 3241) \cup \mathcal{A}(132, 4321, 4213)$ .

**Theorem.** [Atkinson, Beals, Murphy, NR]

$X$  is atomic iff it satisfies the join property:

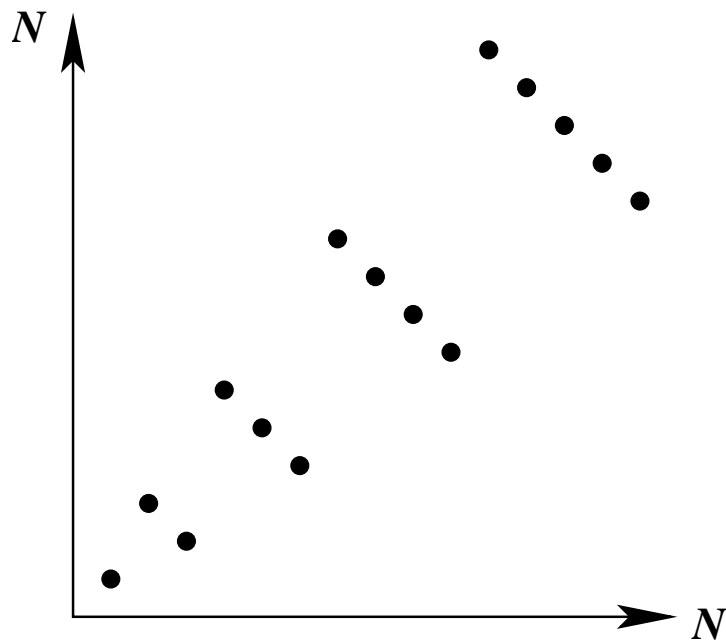
$$\forall \sigma, \tau \in X : \exists \pi \in X : \sigma \preceq \pi \ \& \ \tau \preceq \pi.$$

**Max says:** The author is confident that similar questions regarding intersections [...] are sufficiently easily answered as not to warrant dedicated coverage. The author also awaits questions about intersections that are worthy of attention!

## Sub() Operator

**Definition.** Let  $A, B$  be (usually infinite) linearly ordered sets, and let  $\pi : A \rightarrow B$  be a bijection. For every finite  $C \subseteq A$ ,  $\pi|_C$  is order isomorphic to a permutation. The set of all such permutations is denoted by  $\text{Sub}(\pi : A \rightarrow B)$  or just  $\text{Sub}(\pi)$ .

**Example.** The class  $L$  of layered permutations represented as  $\text{Sub}(\pi : \mathbb{N} \rightarrow \mathbb{N})$ :



## Sub() Operator

**Theorem.** [Atkinson, Beals, Murphy, NR]

$X$  is an atomic closed class iff

$X = \text{Sub}(\pi : A \rightarrow B)$  for some  $A, B$  (which may be chosen to be finite or countable) and  $\pi$ .

## Natural Classes

**Definition.** A closed class is natural if it can be represented as  $\text{Sub}(\pi : \mathbb{N} \rightarrow \mathbb{N})$ .

**Example.** The class  $L$  of layered permutations is natural.

**Question.** Is it decidable whether a finitely based class is natural?

## Direct Sum

**Definition.** For  $\sigma = s_1 \dots s_m$  and  $\tau = t_1 \dots t_n$  define

$$\sigma \oplus \tau = \begin{array}{|c|} \hline \beta \\ \hline \alpha \\ \hline \end{array} = s_1, \dots, s_m, t_1 + m, \dots, t_n + m.$$

**Example.**  $132 \oplus 213 = 132546$ .

**Definition.**  $X \oplus Y = \{\sigma \oplus \tau : \sigma \in X, \tau \in Y\}$ .

**Fact.**  $X, Y$  closed  $\Rightarrow X \oplus Y$  closed.

**Examples.**

For  $I = \{1, 12, 123, \dots\}$  we have  $I \oplus I = I$ .

For  $R = \{1, 21, 321, \dots\}$  we have  
 $R \oplus R = \mathcal{A}(123, 312, 231)$ .

## Sum Completeness

**Definition.**  $X$  is sum complete if  
 $\sigma, \tau \in X \Rightarrow \sigma \oplus \tau \in X.$

**Examples.**  $I$  is sum complete,  $R$  is not.

**Fact.**  $X$  sum complete  $\Rightarrow X$  atomic.  
(Join property:  $\alpha \oplus \beta$  joins  $\alpha$  and  $\beta$ .)

**Fact.**  $X$  sum complete  $\Rightarrow X$  natural.

## Structure of Natural Classes

**Theorem.** [Atkinson, Murphy, NR] If  $X$  is a finitely based natural class then one of the following holds:

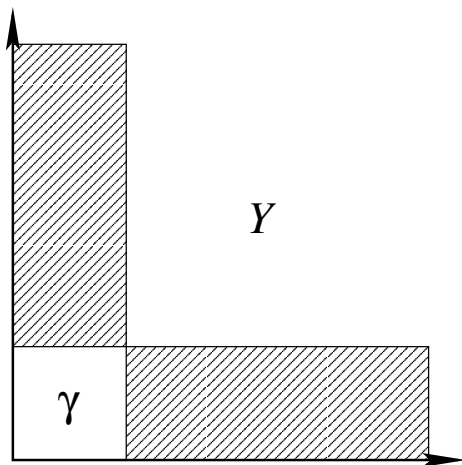
(I)  $X$  is sum complete; or

(II)  $X = \text{Sub}(\gamma) \oplus Y$ , where  $\gamma$  is a (finite) permutation,  $Y$  is sum complete and uniquely determined by  $X$ ; or

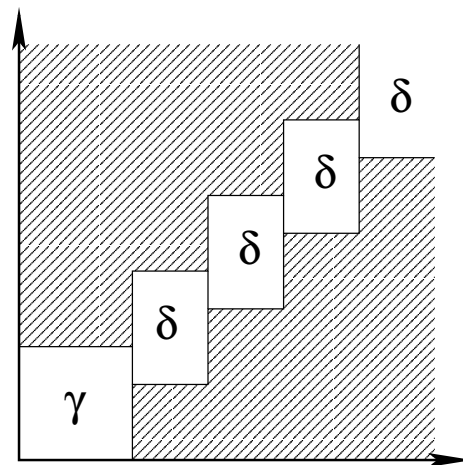
(III)  $X = \text{Sub}(\pi : \mathbb{N} \rightarrow \mathbb{N})$  where  $\pi$  is uniquely determined by  $X$  and is ultimately periodic:

$$\exists N, \omega : \forall n \geq N : \pi(n + \omega) = \pi(n) + \omega.$$

## Natural Classes: Types II and III



Type II



Type III

$\gamma$  – initial segment;

$\delta$  – period.

## Basis $\rightarrow$ Sum Completeness

**Definition.**  $\sigma$  is sum indecomposable if  $\sigma \not\leq \tau \oplus \pi$  for any (non-empty)  $\tau, \pi$ .

**Theorem.** A closed class  $X$  is sum complete iff every element of its basis is sum indecomposable. (Sum completeness is decidable from the basis.)

**Proof.** ( $\Rightarrow$ ) Suppose  $\sigma = \tau \oplus \pi \in \mathcal{B}(X)$ . Then  $\tau, \pi \leq \sigma$  implies  $\tau, \pi \in X$ , but  $\tau \oplus \pi \notin X$ .

( $\Leftarrow$ )  $\sigma, \tau \in X$   
 $\Leftrightarrow \forall \beta \in \mathcal{B}(X) : \beta \not\leq \sigma \ \& \ \beta \not\leq \tau$   
 $\Rightarrow \forall \beta \in \mathcal{B}(X) : \beta \not\leq \sigma \oplus \tau$   
 $\Leftrightarrow \sigma \oplus \tau \in X$ .

## Direct Sum $\rightarrow$ Basis

**Open Problem.** Is it decidable whether the direct sum of two finitely based classes is finitely based?

**Example.** [Atkinson, Murphy]  $\mathcal{A}(321654) \oplus \mathcal{A}(321654)$  is not finitely based.

**Theorem.** [Murphy] For any finite permutation  $\gamma$  and any finitely based class  $Y$  the class  $\text{Sub}(\gamma) \oplus Y$  is finitely based (and the basis is effectively computable).

## Sub( $\gamma$ ) $\oplus$ Y

Let  $X = \text{Sub}(\gamma) \oplus Y$  be a finitely based natural class of type (II) and

$B :=$ (finite) basis of  $X$ ,

$b := \max\{|\beta| : \beta \in B\}$ ,

$C :=$ minimal elements of  $\{\gamma : \gamma \text{ is a final sum component of some } \beta \in B\}$ ,

$c := \max\{|\tau| : \tau \in C\}$ .

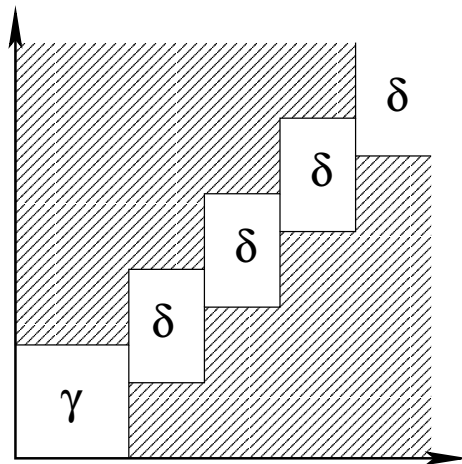
**Theorem.**  $C$  is the basis for  $Y$ .

**Theorem.** There exists  $\gamma_1$  such that  $X = \text{Sub}(\gamma_1) \oplus Y$  and

$$|\gamma_1| \leq c + ((c + 1)^2 + 2b - 1)(b - 1)^2 = F(b, c).$$

**Remark.** Spot  $(b - 1)^2!$  It reflects burried applications of Erdős–Szekeres.

## Periodic $\pi$



**Theorem.** There exist effectively computable functions  $G(b, c)$  and  $H(b, c)$  such that  $|\gamma| \leq G(b, c)$  and  $|\delta| \leq H(b, c)$ .

**Remark.** ... with another two appearances of  $(b - 1)^2!$

## Periodic $\pi$

For any initial segment  $\gamma$ , and any period  $\delta$ , and any way of fitting them together to form a natural class  $X$  of type III the following are true:

- Via the rank encoding (Atkinson, Livsey)  $X$  is a regular class in the sense of formal language theory (Murphy, NR).
- An automaton accepting (the encoded form of)  $X$  can be effectively computed (Murphy, NR).
- From this automaton the (finite or infinite) basis of  $X$  can be effectively computed (Albert, Atkinson, NR).

## Basis $\rightarrow$ Natural

**Theorem.** [Murphy, NR] It is decidable whether a finitely based class (given by its basis  $B$ ) is natural.

**ALGORITHM.** Run Algorithms AlgI, AlgII and AlgIII in parallel. If all three stop without saying 'YES' then say 'NO' :-)

In the algorithms below  $C$ ,  $b$ ,  $c$  are as before. Also we let  $Y := \mathcal{A}(C)$ .

### **AlgI.**

IF (every  $\beta \in B$  is sum indecomposable) THEN RETURN 'YES,  $X$  is sum complete'

## Basis $\rightarrow$ Natural

### AlgII.

```
FOR (every  $\gamma$  with  $|\gamma| \leq F(b, c)$ ) DO
   $D := \mathcal{B}(\text{Sub}(\gamma) \oplus Y)$ 
  IF  $D = B$  THEN RETURN 'YES, type II'
END DO
```

### AlgIII.

```
FOR (every  $\gamma$  with  $|\gamma| \leq G(b, c)$ ) DO
  FOR (every  $\delta$  with  $|\delta| \leq H(b, c)$ ) DO
    FOR (every  $\pi$  with initial segment  $\gamma$ 
          and period  $\delta$ ) DO
       $D := \mathcal{B}(\text{Sub}(\pi))$ 
      IF  $D = B$  THEN RETURN 'YES, type III'
    END DO
  END DO
END DO
```

## Open Problems

Is it decidable whether a finitely based class, given by its basis, can be represented as  $\text{Sub}(\pi : A \rightarrow B)$  where

- (i)  $A = \mathbb{N} \oplus \mathbb{N}$ ,  $B = \mathbb{N}$ ?    (ii)  $A = \mathbb{Z}$ ,  $B = \mathbb{N}$ ?
- (iii)  $A = B = \mathbb{Z}$ ?                      (iv)  $A = B = \mathbb{Q}$ ?
- (v)  $A$  and  $B$  are well ordered? ...

**And finally, Max says:** What we are writing is not rigorous, but we feel that the situation is, like many things, aptly described by Bagehot in “The English Constitution” when he states that it is not wrong to yearn for something before it is possible to achieve it, indeed that often the yearning is an essential prerequisite to the achieving. This was in discussing the early, crushed, uprisings that eventually led to the liberation and unification of Italy in 1852. We believe that an attack on fundamental antichains will lead to classification in terms of the three points mentioned above.

# Algorithms

Algorithm

= Turing machine

= effective computation

= computer program

= unambiguous recipe

- know one when you see it;
- approach borrowed from algebra;
- a fresh look at the area;
- ignore technical issues: complexity, efficiency, etc (for the moment);
- computational tools would be useful nevertheless.

## Permutations as Patterns

Sequence = a finite list of distinct numbers.

Permutation = a sequence of numbers  $1, \dots, n$  of length  $n$ .

Order isomorphism: for sequences  $\sigma = s_1 \dots s_m$ ,  $\tau = t_1 \dots t_n$  we say

$$\sigma \cong \tau \Leftrightarrow m = n \ \& \ (\forall i, j : s_i \leq s_j \Leftrightarrow t_i \leq t_j).$$

**Example.**  $8463 \cong 4231$ ,  $8463 \not\cong 4321$ .

**Fact.** For every sequence  $\sigma$  there is a unique permutation  $\bar{\sigma}$  such that  $\sigma \cong \bar{\sigma}$ .

**Example.**  $\overline{8463} = 4231$ .

## Involvement and Avoidance

**Definition.**  $\sigma = s_1 \dots s_m$  is involved in  $\tau = t_1 \dots t_n$  if  $\tau$  contains a subsequence order isomorphic to  $\sigma$ :

$$\sigma \preceq \tau \Leftrightarrow \exists 1 \leq i_1 < i_2 < \dots < i_m \leq n : t_{i_1} \dots t_{i_m} \cong \sigma.$$

If  $\sigma \not\preceq \tau$  we say that  $\tau$  avoids  $\sigma$ .

**Example.**  $123 \preceq 32415$ ,  $123 \not\preceq 35421$ .

**Fact.**  $\preceq$  is a preorder (RT) on the set  $T$  of all sequences, and is a (partial) order (RAST) on the set  $S$  of all permutations.

**Fact.**  $(T, \preceq) / \{(\sigma, \tau) : \sigma \preceq \tau \ \& \ \tau \preceq \sigma\} \cong (S, \preceq)$ .

# Closed (=Pattern Avoidance) Classes

**Definition.** A set  $X \subseteq S$  of permutations is said to be closed if

$$\sigma \in X \ \& \ \tau \preceq \sigma \Rightarrow \tau \in X.$$

Another name: pattern avoidance class (why: see later).

## Examples.

- $S$ ;
- $I = \{1, 12, 123, \dots\}$ ;
- $R = \{1, 21, 321, \dots\}$ ;
- if  $X$  and  $Y$  are closed then so is  $X \cup Y$ .

## Basis

**Definition.** The basis  $\mathcal{B}(X)$  of a closed class  $X$  is the set of minimal elements not in  $X$ :

$$\mathcal{B}(X) = \{\sigma \notin X : (\forall \tau)(\tau \prec \sigma \Rightarrow \tau \in X)\}.$$

**Example.**  $\mathcal{B}(S) = \emptyset$ ;  $\mathcal{B}(I) = \{21\}$ ;  $\mathcal{B}(I \cup R) = \{132, 213, 231, 312\}$ .

**Definition.** For a set  $Z \subseteq S$  its avoidance set is

$$\mathcal{A}(Z) = \{\sigma : (\forall \tau \in Z)(\tau \not\prec \sigma)\}.$$

### Facts.

- $\mathcal{B}(X)$  is an antichain.
- $\mathcal{A}(Z)$  is closed.
- $\mathcal{A}(\mathcal{B}(X)) = X$ .

## Enumeration

$s_n(X)$  = the number of permutations of length  $n$  in  $X$ .

**Example.**  $s_n(I) = s_n(R) = 1$ ;  $s_n(R \oplus R) = n$ .

**Theorem.** [Knuth; Simion, Schmidt] If  $\pi$  is a permutation of length 3 then  $s_n(\mathcal{A}(\pi)) = C_n$ , the  $n$ th Catalan number.

**Remarks.** Exact formulae are known for  $s_n(\mathcal{A}(\pi))$  where  $\pi$  is any permutation of length 4, except when  $\pi$  is (equivalent to) 1324. Very little is known about  $s_n(\mathcal{A}(\pi))$  when  $|\pi| \geq 5$ .

## Enumeration: Generating Functions

**Open Questions.** Is it decidable whether the ordinary generating function of  $s_n(X)$  for a finitely based closed class  $X$  is (a) rational? (b) algebraic? (c)  $D$ -finite?

**Theorem.** [Bousquet-Melou] The generating function for  $\mathcal{A}(1234)$  is  $D$ -finite but not algebraic.

**Question.** [Gessel] Is the generating function for a finitely based closed class always  $D$ -finite?

**Theorem.** [Murphy] There exists an infinitely based closed class the generating function of which is not  $D$ -finite.

## Enumeration: Growth

**Theorem** . If  $X \neq S$  is a closed class then there exists  $q$  such that  $s_n(X) \leq q^n$ .

**Remarks**. Proved very recently by Marcus and Tardos. Before that it was known as the Wilf–Stanley Conjecture.

**Corollary**. The limit  $q = \lim_{n \rightarrow \infty} \sqrt[n]{s_n(X)}$  exists. (The growth of  $X$ .)

**Examples**. The growth of  $\mathcal{A}(\pi)$  where  $|\pi| = 3$  is 4. [Regev] The growth of  $\mathcal{A}(12 \dots k)$  is  $(k - 1)^2$ .

**Questions**. For a fixed  $q$ , is it decidable whether  $q$  is the growth of a finitely based class  $X$ ? Is the growth of a finitely based class effectively computable?

**Remark**. [Bona] The growth of  $\mathcal{A}(12453)$  is  $9 + 4\sqrt{2}$ .

## Token Passing Networks (TPNs)

A TPN is a finite directed graph with a distinguished input vertex  $I$  and a distinguished output vertex  $O$ . Each vertex is one of the following:

- simple node, capable of holding one item of data;
- stack, capable of holding any number of items, and treating them in the FILO discipline;
- queue (FIFO);
- ...

If all vertices are of type (i) we have a finite capacity (FC) TPN.

## TPNs

Let  $N$  be a TPN.  $N$  can generate permutations: feed  $12\dots n$  into  $N$ , item by item, through  $I$ , move items along edges respecting the orientation, store them in vertices respecting the type, output them via  $O$ .

$\mathcal{P}(N) =$  the set of all output permutations.

**Facts.**  $\mathcal{P}(N)$  is closed, sum complete and atomic.

**Remark.** Now atomicity is decidable!

## TPN: Example

$$X = \mathcal{P}(N)$$

- $X$  is not finitely based.
- Unknown:  $\mathcal{B}(X)$ ;  $s_n(X)$ ; growth; enumeration for the basis.

## FCTPNs

Let  $N$  be a FCTPN,  $|N| = m$ ,  $X = \mathcal{P}(N)$ .

**Theorems.** (i) [Atkinson, Livsey, Tulley] Elements of  $N$  can be encoded, symbol by symbol, by words over an alphabet of size at most  $m$ . [The actual size of the alphabet will be called the boundedness of  $X$ .] The resulting set of words is a regular language.

(ii) The generating function for  $X$  is rational and can be effectively computed.

(iii) [Albert, Atkinson, NR]  $\mathcal{B}(X)$  can be encoded in the same fashion, over the same alphabet, again yielding a regular language.

(iv) It is decidable whether  $X$  is finitely based.

**Remark.** The above algorithms are practical and have been implemented in GAP.

## FCTPNs

**Theorem.** [Albert, Linton, NR] For any  $m$  there are only finitely many pattern classes of the form  $\mathcal{P}(N)$ , where  $N$  is a FCTPN of bound-  
edness  $m$ .

**Fact.** If a class  $X$  is given by its basis, it is decidable whether  $X$  is bounded.

**Theorem.** It is decidable whether a given finitely based class is generated by a FCTPN.

## TPNs

**Question.** Is it decidable whether a given TPN generates a finitely based class?

**Theorem.** [Waton] It is decidable whether a given TPN generates  $S$ , the class of all permutations.

**Idea.** Avoid

# Partial Well Order: Towards Undecidability?

**Definition.** [Following G. Higman] A pattern class  $X$  is said to be partially well ordered (PWO) if it contains no infinite antichain.

**Theorem.** [Atkinson, Murphy, NR] A finitely based closed class is PWO iff it has only countably many subclasses.

## Examples.

- $\mathcal{A}(12)$  is PWO.
- [Atkinson, Murphy, NR]  $\mathcal{A}(231)$  is PWO.
- [Spielman, Bona]  $\mathcal{A}(123)$  is not PWO.

**Questions.** Is partial well orderedness decidable for closed classes given by (a) finite bases? (b) TPNs? (c) FCTPNs?

```
F:=FUNCTION(list)
  RETURN true;
END FUNCTION
```