Some Combinatorial Properties of Direct Products of Groups, Semigroups and Other Algebraic Structures

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University
of
St Andrews

Preview: $1 + 1 = 2$

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Instead of an Introduction: $d(A_n)$, $d(S_n)$

Definition

The rank of an algebraic structure is the smallest number of generators needed to generate A; it is denoted d(A).

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Facts

$d(A_5^{19}) = 2$, $d(A_5^{20}) = 3$.

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$$A \xleftarrow{\pi_A : (a,b) \longrightarrow a} A \times B \xrightarrow{\pi_B : (a,b) \longrightarrow b} B$$

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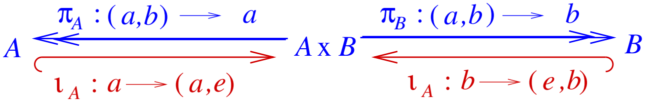

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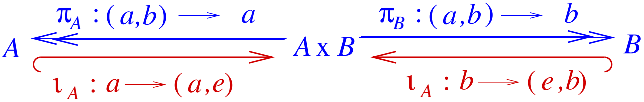

Provided e is an idempotent

Nice, Boring Theorems

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Let G and H be two groups, and let $\mathcal{P}$ be any of the following properties: being finite, abelian, soluble, nilpotent, periodic, finitely generated, finitely presented, locally finite, residually finite, . . .

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Remark

This works for monoids.

Growth of Direct Powers

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Corollary

For any monoids M, N we have

$$\max(d(M), d(N)) \leq d(M \times N) \leq d(M) + d(N).$$

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For a monoid M we have

$$d(M) \leq d(M^n) \leq nd(M).$$

Growth Sequences: Finite Groups

J. Wiegold (with J.S. Wilson, D. Meier, A.G.R. Stewart, A. Efranian), 1974–1995.

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In the non-monoid case, is $\mathbf{d}(S)$ eventually exponential?

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Does such a monoid exist?

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Proposition

For every $k \geq 1$ there exist $a_1, \ldots, a_k \in T_{\mathbb{N}}$ such that

$$\underbrace{T_{\mathbb{N}} \times \ldots T_{\mathbb{N}}}_{k} = (a_1, \ldots, a_k) T_{\mathbb{N}}.$$

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The monoid R of all recursive functions $\mathbb{N} \to \mathbb{N}$ is finitely generated, and also has the above property.

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Remark

Neither $T_{\mathbb{N}}$ nor R are congruence-free.

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Let M, N be monoids. $M \times N$ is finitely generated if and only if M and N are finitely generated.

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Finite Generation: Semigroups

Theorem (EF Robertson, NR, J. Wiegold)

Let S, T be infinite semigroups. $S \times T$ is finitely generated if and only if

(i) *S and T are finitely generated; and*

(ii) *neither S nor T have indecomposable elements.*

Theorem

Let S, T be semigroups, with S infinite, T finite. $S \times T$ is finitely generated if and only if

(i) *S is finitely generated; and*

(ii) *T has no indecomposable elements.*

Finite Presentability

Theorem

Let M, N be monoids. $M \times N$ is finitely presented if and only if M and N are finitely presented.

Example

$\mathbb{N}$ is finitely presented (in fact free), but $\mathbb{N} \times \mathbb{N}$ is not finitely presented.

Question

Will $S \times T$ be finitely presented provided S and T are finitely presented and $S \times T$ is finitely generated?

Critical Pairs and Stability

S – a semigroup; $\langle A|R\rangle$ a finite presentation for it.

Fact

Two words u, v over A are equal in S if and only if there is a sequences of applications of relations from R (a deduction) which transforms u into v.

Definition

A pair (u, v) of words is critical if every deduction from u to v contains a word of length smaller than $\min(|u|, |v|)$.

Definition

S is said to be stable if it has no critical pairs.

Remark

The above definition of stability is not constructive.

Stability and Finite Presentability

Theorem (EF Robertson, NR, J Wiegold)

Let S, T be two infinite semigroups. $S \times T$ is finitely presented if and only if

(i) *S and T are (finitely presented) and stable; and*

(ii) *neither S nor T contain indecomposable elements.*

Theorem (EF Robertson, NR, J Wiegold)

Let S be an infinite semigroup, and let T be a finite semigroup. $S \times T$ is finitely presented if and only if

(i) *S is finitely presented; and*

(ii) *T is stable and contains no indecomposable elements.*

Some Non-Finitely-Presented Examples

Example (Araujo, NR)

The four element semigroup

S	a	b	c	0
a	a	a	c	0
b	b	b	c	0
c	0	0	0	0
0	0	0	0	0

is a non-stable semigroup of minimal size.

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Equivalently: for any two $a, b \in A$ ($a \neq b$) there exists a congruence ρ with finitely many classes such that $(a, b) \notin \rho$.

Residual Finiteness: Definition

Definition

An algebraic structure A is residually finite if for any two $a, b \in A$ ($a \neq b$) there is a homomorphism $f : A \to B$, B finite, such that $f(a) \neq f(b)$.
Equivalently: for any two $a, b \in A$ ($a \neq b$) there exists a congruence ρ with finitely many classes such that $(a, b) \notin \rho$.
Equivalently: the intersection of all finite index congruences is trivial.

Residual Finiteness: General, Nice, Boring Theorem?

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Proof

If $e \in A$ is an idempotent, then $B \cong \{e\} \times B \leq A \times B$.

Residual Finiteness: Semigroups

R. Gray, NR

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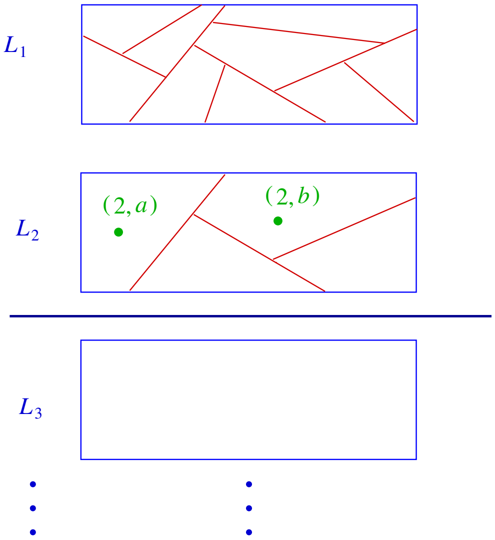
Intersect ρ and λ to obtain a congruence $\sigma = \rho \cap \lambda$ which has finitely many classes, respects levels 1 and 2, and separates $(2, a)$ and $(2, b)$.

Residual Finiteness: Levels of $\mathbb{N} \times S$



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Theorem (Gray, NR)

Let S and T be semigroups. $S \times T$ is residually finite if and only if S and T are residually finite.

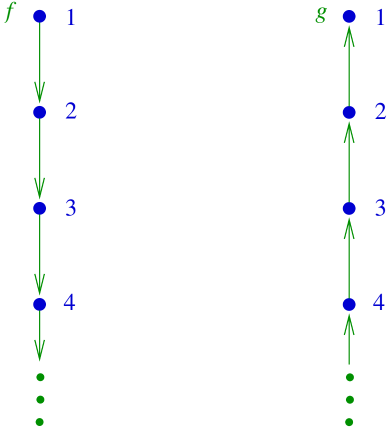
Proof

A semigroup either contains an idempotent or a copy of $\mathbb{N}$:-)

Residual Finiteness: Unary Algebras

Example

Consider two unary operations on $\mathbb{N}$:



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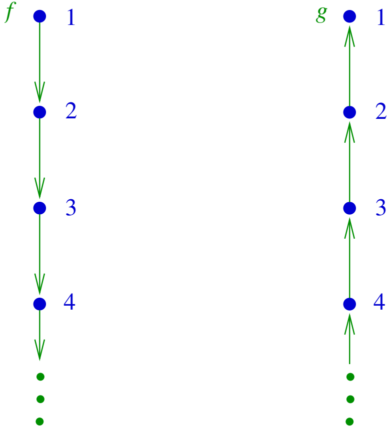

Let $A = (\mathbb{N}, f)$, $B = (\mathbb{N}, g)$.

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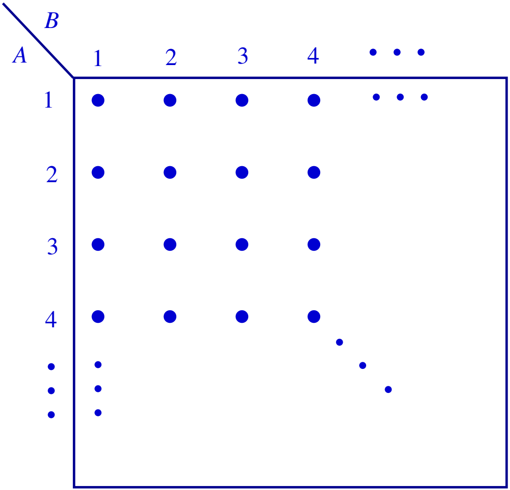

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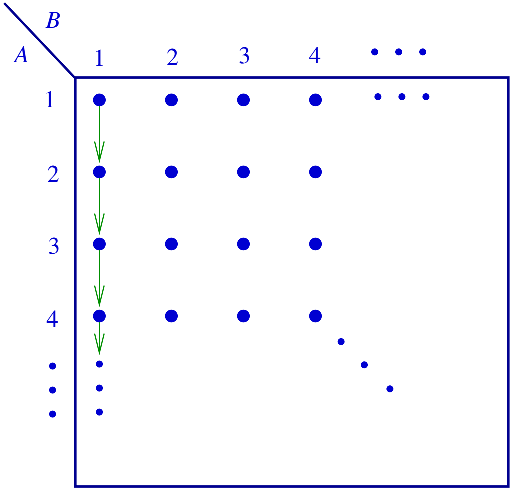

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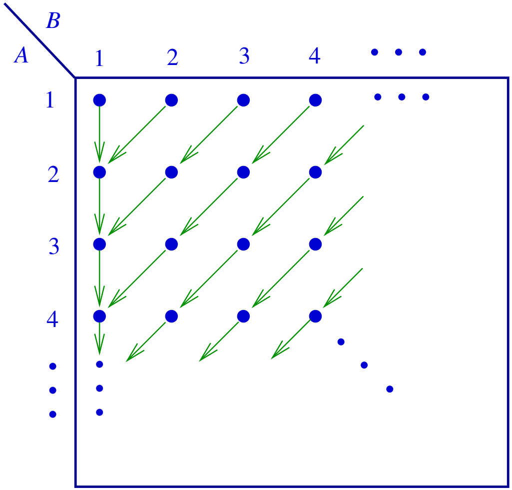

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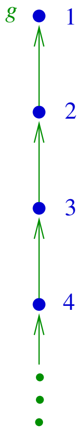

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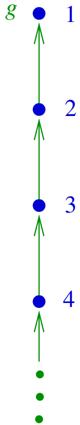

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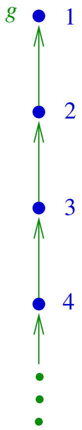

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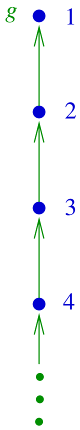

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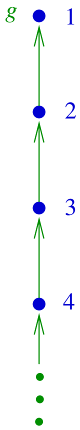

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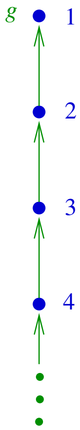

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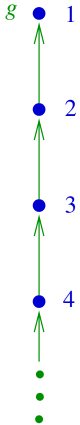

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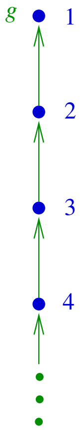

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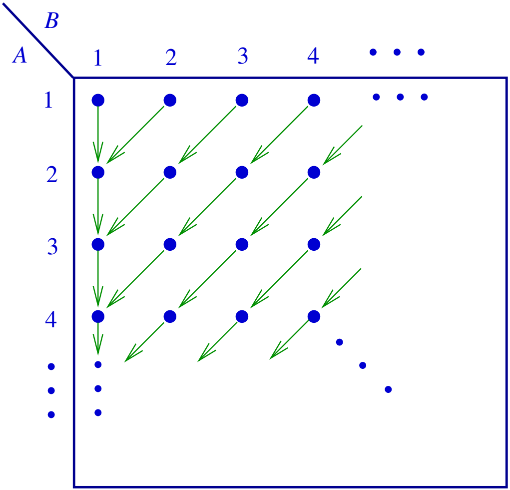

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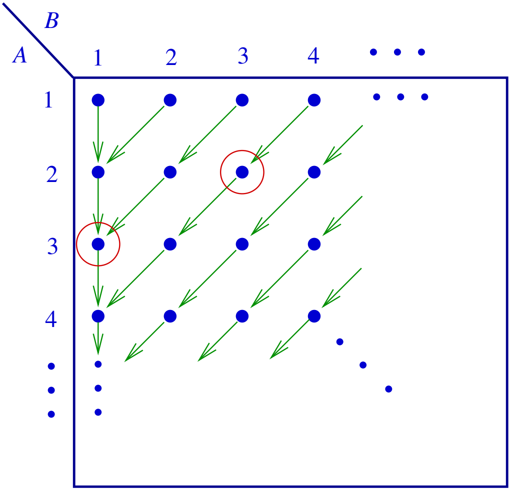

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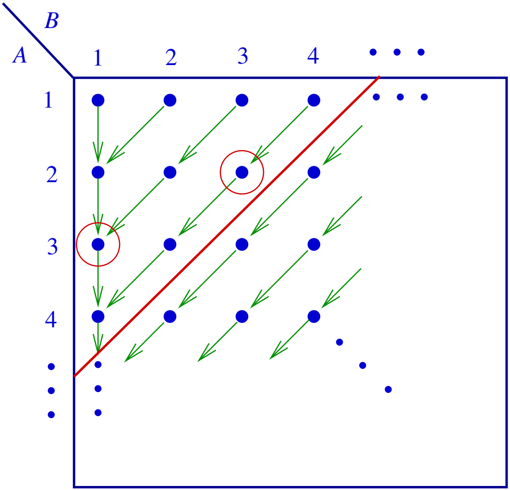

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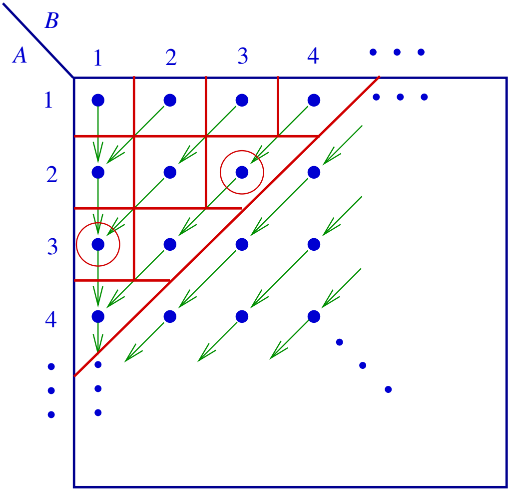

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Other products: wreath product, work in progress with M Quick, M Neunhöffer.