Growth of Generating Sets of Direct Powers

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Glasgow, 30 September 2009

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Let S and T be two infinite semigroups. The direct product $S \times T$ is finitely presented if and only if it is finitely generated

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Theorem. (E.F. Robertson, NR, J. Wiegold, 1998)
Let S and T be two infinite semigroups. The direct product $S \times T$ is finitely presented if and only if it is finitely generated and both S and T are stable.

A mini-quiz: $d(A_5)$, $d(S_5)$

Definition

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$d(A_5^{19}) = 2$, $d(A_5^{20}) = 3$.

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The **d**-sequence

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The *d*-sequence of an algebraic structure A is

$$\mathbf{d}(A) = (d(A), d(A^2), d(A^3), \ldots).$$

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Some basic properties:

- $\mathbf{d}(A)$ is non-decreasing.
- $\mathbf{d}(A)$ is bounded above by $|A|^n$.
- Often $d(A \times B) \leq d(A) + d(B)$, in which case $\mathbf{d}(A)$ is bounded above by a linear function.

Pozzo and Vladimir Introduce Types of Growth

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- Zero semigroup: $\mathbf{d}(Z_2) = (1, 3, 7, 15, \ldots)$ (poor).

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J Wiegold: **d**-sequences of groups

- J. Wiegold, Growth sequences of finite groups 1-5 (5 with D. Meier), J. Austral Math. Soc. 1974–1981.
- J. Wiegold and J.S. Wilson, Growth sequences of finitely generated groups, Arch. Math. (Basel) 1978.
- A.G.R. Stewart and J. Wiegold, Growth sequences of finitely generated groups II, Bull. Austral. Math. Soc. 1989.

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- $\mathbf{d}(G)$ is bounded above by a logarithmic function if G is infinite and perfect.

$\mathbf{d}(G^n) \leq d(G) + 1$ (G infinite simple)

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Theorem (MR Quick, NR)

If A is finite functionally complete then $\mathbf{d}(A)$ is logarithmic.

Functionally complete classical structures

Definition
Classical structures: groups, rings, modules, algebras, Lie algebras.

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Functionally complete finite classical structures are: non-abelian simple groups, simple rings with identity, simple algebras with identity, non-abelian simple Lie algebras.

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Corollary

All of the above have logarithmic $\mathbf{d}$*-sequences.*

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A finite simple algebraic structure A in a congruence permutable equational class is either:

- *functionally complete, in which case $\mathbf{d}(A)$ is logarithmic; or*
- *polynomially equivalent to a simple module over a finite ring with 1, and $\mathbf{d}(A)$ is linear.*

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Remark

Jump from simple to arbitrary requires more work, and a generalisation of a lovely old trick of Gaschütz, for lifting generating sets to pre-images.

Infinite classical structures

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- At worst: linear.

Question

Is the identity element necessary for a good growth? Does there exist a finitely generated infinite simple ring without identity?

Wiegold on finite semigroups

J. Wiegold, Growth sequences of finite semigroups, J. Austral. Math. Soc. 1987.

Theorem

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Theorem

For a finite (non-group) semigroup S we have:

- $\mathbf{d}(S)$ *is linear if* S *is a monoid.*
- *otherwise* $\mathbf{d}(S)$ *is exponential.*

Polycyclic monoid

Definition

$$P_k = \langle b_i, c_i \ (i = 1, \dots, k) \mid b_i c_i = 1,\ b_i c_j = 0 \ (i \neq j) \rangle$$

Fact

P_k ($k \geq 2$) is an infinite, congruence-free monoid.

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Theorem (St Andrews Summer School 2008)

$\mathbf{d}(P_k) = (2k - 1, 3k - 1, 4k - 1, \ldots)$.

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Theorem (EF Robertson, NR, J Wiegold)

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$$d(S \times T) \leq 4d(S)d(T).$$

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Corollary

If $d(S^2) < \infty$ then all S^n are finitely generated, and $\mathbf{d}(S)$ grows exponentially at worst.

Cyclic diagonal acts

Definition

A semigroup S is said to have cyclic diagonal bi-acts if for every $n \geq 1$ there exist $a_1, \ldots, a_n \in S$ such that

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and so $\mathbf{d}(S)$ is eventually constant.

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Theorem (St Andrews Summer School 2008)

For the monoid $R_{\mathbb{N}}$ of all partially recursive functions in one variable we have

$$\mathbf{d}(R_{\mathbb{N}}) = (2, 2, 2, \dots).$$

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$A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$ – two infinite alphabets.

Let $P = (p_{ab})_{A\times B}$ be a matrix over $A \cup B \cup \{0\}$ such that:

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Define

$$S = \langle A, B \mid ab = p_{a,b} \rangle.$$

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There exists an infinite semigroup S without identity for which $\mathbf{d}(S)$ is (a) logarithmic; (b) linear.

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Theorem (MR Quick, NR)

R is a finitely generated infinite simple ring without identity, and $\mathbf{d}(R)$ is eventually constant.

Two-element structures

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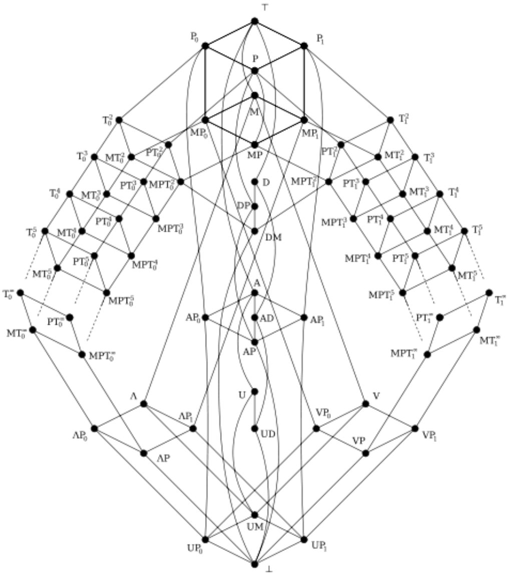
Remark (Agoston et al. 1986)

There are uncountably many inequivalent algebraic structures on a 3-element set.

Post's Lattice



A selection of problems

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- Does there exist an infinite simple group G and $n > 1$ such that $d(G^n) \neq d(G)$. (*One wonders whether or not these results reflect a general truth about infinite simple groups.*, J Wiegold 1978)

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- Number of relations; higher homological invariants.