1 + 1 = 2 makes direct products interesting

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... for I cannot satisfy myself that, when one is added to one, the one to which the addition is made becomes two, or that the two units added together make two by reason of the addition. I cannot understand how, when separated from the other, each of them was one and not two, and now, when they are brought together, the mere juxtaposition or meeting of them should be the cause of their becoming two: neither can I understand how the division of one is the way to make two; for then a different cause would produce the same effect,—as in the former instance the addition and juxtaposition of one to one was the cause of two, in this the separation and subtraction of one from the other would be the cause. Nor am I any longer satisfied that I understand the reason why one or anything else is either generated or destroyed or is at all, but I have in my mind some confused notion of a new method... (Socrates in Plato’s Phaedo)
The generic problem

\( \mathcal{P} \) – an algebraic property (finiteness condition).

**Generic Problem**
Find a necessary and sufficient condition (in terms of \( A \) and \( B \)) for the direct product \( A \times B \) to satisfy property \( \mathcal{P} \).

**Generic Theorem**
\( A \times B \) satisfies \( \mathcal{P} \) iff \( A \) and \( B \) satisfy \( \mathcal{P} \).

**Definition**
NICE, BORING THEOREM.

**Examples**
Finiteness, periodicity (for semigroups), finite generation (for groups and monoids).
$A, B, A \times B$
$A, B, A \times B$

\[\pi_A : (a,b) \rightarrow a\]

\[\pi_B : (a,b) \rightarrow b\]

\[A \leftarrow A \times B \rightarrow B\]
$A, B, A \times B$

\[ \pi_A : (a,b) \rightarrow a \quad \pi_B : (a,b) \rightarrow b \]

\[ A \quad \leftrightarrow \quad A \times B \quad \leftrightarrow \quad B \]
$A, B, A \times B$

\[\begin{align*}
\pi_A : (a,b) &\to a \\
\iota_A : a &\to (a,e)
\end{align*}\]

\[\begin{align*}
\pi_B : (a,b) &\to b \\
\iota_A : b &\to (e,b)
\end{align*}\]
$A, B, A \times B$

$\pi_A : (a, b) \rightarrow a$

$\pi_B : (a, b) \rightarrow b$

$\iota_A : a \rightarrow (a, e)$

$\iota_A : b \rightarrow (e, b)$

Provided $e$ is an idempotent
Example
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Note: $(1, n) \neq (a, b) + (c, d)$.
Because: $a + c \geq 1 + 1 = 2 > 1$:(-)
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We say that $(1, n)$ is indecomposable.
Example

Consider the additive semigroup $\mathbb{N} \times \mathbb{N}$ ($0 \notin \mathbb{N}$).

Note: $(1, n) \neq (a, b) + (c, d)$.

Because: $a + c \geq 1 + 1 = 2 > 1$.

We say that $(1, n)$ is indecomposable.

So, $\{(1, n) : n \in \mathbb{N}\}$ is contained in every generating set.
Finite Generation: Semigroups

Theorem (EF Robertson, NR, J. Wiegold)

Let $S$, $T$ be infinite semigroups.

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Let $S, T$ be infinite semigroups. $S \times T$ is finitely generated if and only if

(i) $S$ and $T$ are finitely generated;
(ii) neither $S$ nor $T$ have indecomposable elements ($SS = S$, $TT = T$).
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Theorem
Let $S$, $T$ be semigroups, with $S$ infinite, $T$ finite. $S \times T$ is finitely generated if and only if

(i) $S$ is finitely generated; and
(ii) $T$ has no indecomposable elements.
Finite Presentability for Semigroups

Question
When is the direct product $S \times T$ of two semigroups finitely presented?

Remarks
- NBT: groups, monoids.
- $S$, $T$ infinite $\Rightarrow$ no indecomposable elements.
Critical Pairs and Stability

$S$ – a semigroup; $\langle A|R \rangle$ a finite presentation for it.

**Fact**

Two words $u, v$ over $A$ are equal in $S$ if and only if there is a sequences of applications of relations from $R$ (a deduction) which transforms $u$ into $v$.

**Definition**

A pair $(u, v)$ of words is critical if every deduction from $u$ to $v$ contains a word of length smaller than $\min(|u|, |v|)$.

**Definition**

$S$ is said to be stable if it has no critical pairs.

**Remarks**

- Definition independent of $A$, but dependent on $R$.
- Not constructive.
Stability and Finite Presentability

Theorem (EF Robertson, NR, J Wiegold)
Let $S, T$ be two infinite semigroups. $S \times T$ is finitely presented if and only if

(i) $S$ and $T$ are (finitely presented) and stable; and
(ii) neither $S$ nor $T$ contain indecomposable elements.

Theorem (EF Robertson, NR, J Wiegold)
Let $S$ be an infinite semigroup, and let $T$ be a finite semigroup. $S \times T$ is finitely presented if and only if

(i) $S$ is finitely presented; and
(ii) $T$ is stable and contains no indecomposable elements.
Corollary

Suppose that $S$ and $T$ belong to any of the following classes: monoids (including groups), regular semigroups (including inverse semigroups), surjective commutative semigroups (…) Then $S \times T$ is finitely presented if and only if $S$ and $T$ are finitely presented.
Some Non-Finitely-Presented Examples

Theorem (I Araujo, NR)

There is an (effective) algorithm which decides whether a finite semigroup is stable.

Example

The four element semigroup

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is a non-stable semigroup of minimal size. Hence, for example, \( S \times \mathbb{Z} \) is finitely generated but not finitely presented.
Definition
A (universal) algebra \((A, F)\) is idempotent if \(f(a, \ldots, a) = a\) for all \(f \in F\) and all \(a \in A\).
Idempotent Algebras

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Lattices are idempotent.
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Theorem (Mayr)

The direct product \(A \times B\) of two idempotent algebras is finitely generated iff \(A\) and \(B\) are finitely generated.
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A (universal) algebra \((A, F)\) is idempotent if \(f(a, \ldots, a) = a\) for all \(f \in F\) and all \(a \in A\).

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Lattices are idempotent.

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The direct product \(A \times B\) of two idempotent algebras is finitely generated iff \(A\) and \(B\) are finitely generated.

Question
What about finite presentability?
Malcev Algebras

Definition
A (universal) algebra \((A, F)\) is Malcev if it possesses a ternary term \(m(x, y, z)\) such that

\[ m(x, y, y) = m(y, y, x) = x. \]

Remark
Groups, rings, modules, associative and Lie algebras are all Malcev.

Theorem (Geddes)
The direct product \(A \times B\) of two Malcev algebras is finitely generated iff \(A\) and \(B\) are finitely generated.

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A (universal) algebra \((A, F)\) is **Malcev** if it possesses a ternary term \(m(x, y, z)\) such that

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Residual Finiteness: Definition

Definition
An algebraic structure $A$ is residually finite if for any two $a, b \in A$ ($a \neq b$) there is a homomorphism $f : A \to B$, $B$ finite, such that $f(a) \neq f(b)$.

Equivalently, $A$ is residually finite if for any two $a, b \in A$ ($a \neq b$) there exists a congruence $\rho$ with finitely many classes such that $(a, b) \notin \rho$. 
Residual Finiteness: (Very) General, Nice, (Very) Boring Theorem?

Proposition

If A and B are residually finite then $A \times B$ is residually finite.

Proof

Take $(a, b) \neq (c, d)$.

Wlog suppose $a \neq c$.

Map: $A \times B \xrightarrow{\pi} A \xrightarrow{f} C$, $C$ finite, $f(a) \neq f(c)$.

Fact

Residual finiteness is a hereditary property: If A is residually finite and $B \leq A$ the A is residually finite as well.

Proposition

Suppose both A and B contain idempotents. Then $A \times B$ is residually finite iff A and B are residually finite.
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If $A$ and $B$ are residually finite then $A \times B$ is residually finite.
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University of St Andrews Nik Ruškuc: 1+1=2
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**Proposition**

*If A and B are residually finite then A × B is residually finite.*

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Map: \(A \times B \xrightarrow{\pi_A} A \xrightarrow{f} C, \ C \text{ finite}, \ f(a) \neq f(c)\).

**Fact**

*Residual finiteness is a hereditary property: If A is residually finite and B \(\leq A\) the A is residually finite as well.*

**Proposition**

*Suppose both A and B contain idempotents. Then A × B is residually finite iff A and B are residually finite.*
Residual Finiteness: Semigroups

R. Gray, NR

Lemma

Let $S$ be a semigroup. If $N \times S$ is residually finite then $S$ is residually finite.

Proof

Let $a, b \in S$, $a \neq b$.

Let $\rho$ be a congruence on $N \times S$ with finitely many classes that separates $(2, a)$ and $(2, b)$.

$N \times S$ naturally splits into levels: $L_i = \{i\} \times S$.

The equivalence relation $\lambda$ with classes $L_1, L_2, L_3 \cup L_4 \cup ...$ is a congruence with finitely many classes.

Intersect $\rho$ and $\lambda$ to obtain a congruence $\sigma = \rho \cap \lambda$ which has finitely many classes, respects levels 1 and 2, and separates $(2, a)$ and $(2, b)$. 
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Residual Finiteness: Levels of $\mathbb{N} \times S$
Residual Finiteness: Semigroups

Proof (contd.)

\[\gamma\] – the equivalence on \(S\) corresponding to the partition of level 1.
\[\delta\] – the equivalence on \(S\) corresponding to the partition of level 2.
\[\tau\] – the right congruence generated by \(\gamma\).

Claim: \(\gamma \subseteq \tau \subseteq \delta\).

Proof: \(\tau\) is obtained by taking pairs \((xu, yu)\) where \((x, y) \in \gamma\), \(u \in S_{1}\), and closing transitively.

But \((x, y) \in \gamma\) \(\Rightarrow ((1, x), (1, y)) \in \sigma\) \(\Rightarrow ((1, xu), (1, yu)) \in \sigma\) \(\Rightarrow ((2, xu), (2, yu)) \in \sigma\) \(\Rightarrow (xu, yu) \in \delta\).

Hence: \(\tau\) has finitely many classes, and separates \(a\) and \(b\).

A technicality to pass to a congruence.
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Residual Finiteness: Semigroups

Proof (contd.)

γ – the equivalence on $S$ corresponding to the partition of level 1.
δ – the equivalence on $S$ corresponding to the partition of level 2.
τ – the right congruence generated by γ.

Claim: $\gamma \subseteq \tau \subseteq \delta$.

Hence:

τ has finitely many classes, and separates a and b.

A technicality to pass to a congruence.
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Proof: $\tau$ is obtained by taking pairs $(xu, yu), (x, y) \in \gamma, u \in S^1$, and closing transitively. But

$$(x, y) \in \gamma \Rightarrow ((1, x), (1, y)) \in \sigma$$
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Proof: $\tau$ is obtained by taking pairs $(xu, yu), (x, y) \in \gamma, u \in S^1$, and closing transitivity. But

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Hence: $\tau$ has finitely many classes,
Residual Finiteness: Semigroups

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\[\implies ((2, xu), (2, yu)) \in \sigma \implies (xu, yu) \in \delta.\]

Hence: \(\tau\) has finitely many classes, and separates \(a\) and \(b\).
A technicality to pass to a congruence.
Residual Finiteness: A Nice, (Not Boring?) Theorem

**Lemma**

Let $S$ be a semigroup. If $\mathbb{N} \times S$ is residually finite then $S$ is residually finite.
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Let $S$ be a semigroup. If $\mathbb{N} \times S$ is residually finite then $S$ is residually finite.

Theorem (Gray, NR)

Let $S$ and $T$ be semigroups. $S \times T$ is residually finite if and only if $S$ and $T$ are residually finite.

Proof

A semigroup either contains an idempotent or a copy of $\mathbb{N}$. 
Residual Finiteness: Unary Algebras

Example

Consider two unary operations on $\mathbb{N}$:

\[ f(1) = 2, f(2) = 3, f(3) = 4, \ldots \]

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Example

Consider two unary operations on $\mathbb{N}$:

Let $A = (\mathbb{N}, f)$, $B = (\mathbb{N}, g)$. 
Residual Finiteness: Unary Algebras

Form the direct product $A \times B$: 
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Residual Finiteness: Unary Algebras

Lemma

* $B$ is not residually finite.*

\[
\begin{align*}
(1, 0) & \in \rho \\
\ldots & \ldots \\
(2, 0) & \in \rho,
\end{align*}
\]

\[
\begin{align*}
(3, 0) & \in \rho \\
(4, 0) & \in \rho \\
\ldots & \ldots \\
(2, 0) & \in \rho
\end{align*}
\]
Lemma

$B$ is not residually finite.

Proof

Suppose $\rho$ is a congruence with finitely many classes separating 1 and 2.
Residual Finiteness: Unary Algebras

Lemma

$B$ is not residually finite.

Proof

Suppose $\rho$ is a congruence with finitely many classes separating 1 and 2.
Let $(m, n) \in \rho$, $m > n$. 

\[ (m, n) \in \rho \Rightarrow (m-1, n-1) = (g(m), g(n)) \in \rho \Rightarrow \ldots \Rightarrow (2, 1) \in \rho, \text{ a contradiction.} \]
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Proof

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Let $(m, n) \in \rho$, $m > n$.
We have

$(m, n) \in \rho$
Lemma

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Let $(m, n) \in \rho$, $m > n$.
We have

$$(m, n) \in \rho \implies$$
Lemma

\emph{B is not residually finite.}

Proof
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Let \((m, n) \in \rho, \ m > n\).
We have

\[(m, n) \in \rho \quad \Rightarrow \quad (m - 1, n - 1) = (g(m), g(n)) \in \rho\]
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a contradiction.

Lemma

$A$ is residually finite.
Residual Finiteness: Unary Algebras

Lemma

$A \times B$ is residually finite.
Lemma

\( A \times B \) is residually finite.

Proof
Lemma

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Proof
Residual Finiteness: Unary Algebras

Lemma

\[ A \times B \text{ is residually finite.} \]

Proof
Residual Finiteness: Unary Algebras

Lemma

$A \times B$ is residually finite.

Proof

[Diagram showing a grid with points and arrows indicating connections between elements, with some points circled to emphasize certain structures.]
Some open problems

Open Problem
Is the following problem algorithmically decidable: Given two finitely presented semigroups $S = \langle A \mid R \rangle$ and $T = \langle B \mid Q \rangle$, is their direct product finitely presented?

Open Problem
Find three finitely presented monoids $S$, $T$, $U$ such that $S \times T \times U$ contains a subsemigroup which projects onto any pair of factors, but which is not finitely presented.

Open Problem
Is it true that $A \times B$ ($A$, $B$ monoids, or even groups) is automatic if and only if $G$ and $H$ are automatic?

Open Problem
Find a necessary and sufficient condition for the wreath product $A \wr B$ of two monoids (or semigroups) to be residually finite.
I admit that twice two makes four is an excellent thing, but if we are to give everything its due, twice two makes five is sometimes a very charming thing too.
(F. Dostoevsky, Notes from the Underground)
Postscript

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THANK YOU!