Ranks of Semigroups

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...the word “mathematical” stems from the Greek expression *ta mathemata*, which means what can be learned and thus, at the same time, what can be taught; ... 

*Teaching therefore does not mean anything else then to let others learn, that is, to bring one another to learning.*

(M. Heidegger)
John Howie on Generators and Ranks


Products of idempotents in $T_X$: finite $X$


$T_X$ – the full transformation semigroup.

$E_X = E(T_X)$ – the idempotents of $T_X$.

**Theorem**

If $X = \{1, \ldots, n\}$ then

$$\langle E_X \rangle = \text{Sing}_X = \{ \alpha : |\text{im}(\alpha)| < n \}.$$
Products of idempotents in $T_X$: infinite $X$

$S(\alpha) = \{x \in X : x\alpha \neq x\}; \quad s(\alpha) = |S(\alpha)| – \text{shift.}$

$Z(\alpha) = X \setminus X\alpha; \quad z(\alpha) = |Z(\alpha)| – \text{defect.}$

$C(\alpha) = \bigcup\{x\alpha^{-1} : |x\alpha^{-1}| \geq 2\}; \quad c(\alpha) = |C(\alpha)| – \text{collapse.}$

**Theorem**

*If $X$ is infinite then*

$$\langle E_X \rangle = \{\alpha : s(\alpha) < \infty, z(\alpha) > 0\} \cup \{\alpha : s(\alpha) = z(\alpha) = c(\alpha) \geq \aleph_0\}.$$
Vista

- Clear finite/infinite separation;
- numbers of generators;
- lengths of products;
- other semigroups (e.g. $K(n, r)$, $O_X$).
Rank: full transformations

\[ \text{rank}(S) = \min \{|A| : \langle A \rangle = S \}. \]

\[ \text{idrank}(S) = \min \{|A| : A \subseteq E(S), \langle A \rangle = S \}. \]

**Theorem (Gomes, Howie '87)**
\[ \text{rank}(\text{Sing}_n) = \text{idrank}(\text{Sing}_n) = \frac{n(n - 1)}{2}. \]

\[ K(n, r) = \{ \alpha \in T_n : |\text{im}\alpha| \leq r \}. \]

**Theorem (McFadden, Howie '90)**
\[ \text{rank}(K(n, r)) = \text{idrank}(K(n, r)) = S(n, r), \text{ the Stirling number}. \]

**Remark**
\[ S(n, r) = \text{number of } \mathcal{R}-\text{classes in the } \mathcal{D}-\text{class} \{ \alpha : |\text{im}(\alpha)| = r \}. \]
Rank: order preserving transformations


\( O_n = \) order preserving transformations of \( \{1, \ldots, n\} \).

**Theorem**
\[ \text{rank}(O_n) = n, \text{idrank}(O_n) = 2n - 2. \]

**Remark**
The top \( J \)-class of \( O_n \) has \( n - 1 \) \( R \)-classes and \( n \) \( L \)-classes.

\( PO_n = \) partial order preserving transformations of \( \{1, \ldots, n\} \).

**Theorem**
\[ \text{rank}(PO_n) = 2n - 1, \text{idrank}(O_n) = 3n - 2. \]

**Remark**
The top \( J \)-class of \( PO_n \) has \( 2n - 1 \) \( R \)-classes and \( n \) \( L \)-classes.
Often $\text{rank}(S) = \text{rank}(P)$, where $P$ is the principal factor corresponding to a unique maximal $\mathcal{J}$-class. Why?

Often $\text{rank}(P) = \max(|P/\mathcal{L}|, |P/\mathcal{R}|)$. Why?

Sometimes $\text{rank}(S) = \text{idrank}(S)$. Why?

*How to find the rank of a general completely 0-simple semigroup?*
Completely 0-simple semigroups: connected case

$S = \mathcal{M}^0[G; I, \Lambda; P]$ – a Rees matrix semigroup.

Bipartite graph $\Gamma(P)$: $V = I \cup \Lambda$; $i \sim \lambda \iff p_{\lambda i} \neq 0$.

$H = a$ subgroup of $G$ depending on $P$ and $\Gamma$.

$H \neq \langle \{ p_{\lambda i} : \lambda \in \Lambda, \ i \in I \} \setminus \{0\} \rangle$.

Relative rank: $\text{rank}(G : H) = \min \{ A : \langle H \cup A \rangle = G \}$.

**Theorem (NR ’94)**

If $\Gamma$ is connected then

$$\text{rank}(S) = \max(|I|, |\Lambda|, \text{rank}(G : H)).$$

**Corollary**

If $S$ is completely simple then

$$\text{rank}(S) = \max(|I|, |\Lambda|, \text{rank}(G : H)),$$

where $H = \langle p_{\lambda i} : \lambda \in \Lambda, \ i \in I \rangle$. 
Completely 0-simple semigroups: general case

\( S = \mathcal{M}^0[G; I, \Lambda; P] \) – a Rees matrix semigroup.

Connected components: \( I_1 \cup \Lambda_1, \ldots, I_k \cup \Lambda_k \).

\( H_i \) = subgroups of \( G \) depending on \( P \) and \( \Gamma \) (\( i = 1, \ldots, k \)).

Theorem (Gray, NR ’05)

\[
\text{rank}(S) = \max(|I|, |\Lambda|, r + k - 1),
\]

where

\[
r = \min\{\text{rank}(G : \bigcup_{i=1}^{k} g_i^{-1}H_ig_i) : g_i \in G\}.
\]
Vista

- Gray '08: A combinatorial condition (Hall-like) for $\text{rank}(S) = \text{idrank}(S)$ ($S$ completely simple).
- Ideas of connectedness.
- Recent work on free idempotent generated semigroups Gray, Dolinka, NR.
- Semigroups of high rank (Giraldes, Howie '85).
- Different notions of rank (Ribeiro, Howie '99, '00).
Depth in infinite semigroups


**Depth of** $S$ **w.r.t. generating set** $A$

= smallest $n$ such that $S = \bigcup_{i=1}^{n} A^n$.

Recall: $\langle E_X \rangle = F_X \cup (\bigcup_{\aleph_0 \leq m \leq |X|} Q_m)$, where $F = \{\alpha : s(\alpha) < \infty, \ d(\alpha) > 0\}$, $Q_m = \{\alpha : s(\alpha) = z(\alpha) = c(\alpha) = m\}$.

**Theorem**

(i) $\text{Depth}(\langle E_X \rangle) = \text{Depth}(F) = \infty$;

(ii) $\text{Depth}(Q_m) = \text{Depth}(\langle E_X \rangle \setminus F) = 4$. 
Relative ranks in infinite semigroups


**Theorem**

\[ \text{rank}(T_X : S_X) = 2. \]

\( K(\alpha) = \{ x \in X : |x\alpha^{-1}| = |X| \}; \ k(\alpha) = |K(\alpha)| \) – infinite contraction index.

**Theorem**

\( \langle S_X, \mu, \nu \rangle = T_X \) iff \( \mu \) is an injection with \( d(\mu) = |X| \) and \( \nu \) is a surjection with \( k(\nu) = |X| \) (or v.v.).

**Theorem**

\[ \text{rank}(T_X : \langle E(T_X) \rangle) = 2. \]
Relative ranks in infinite semigroups


**Theorem (Sierpiński ’35)**

For any $\tau_1, \tau_2, \cdots \in T_X$ there exist $\mu, \nu \in T_X$ such that $\tau_i \in \langle \mu, \nu \rangle$.

**Corollary**

For every $S \leq T_X$ either $\text{rank}(T_X : S) \leq 2$ or else $\text{rank}(T_X : S)$ is uncountable.

- Analogous results hold for: $S_X$ (Galvin ’95), $B_X$, $I_X$.
- $\text{rank}(B_X : T_X) = \text{rank}(B_X : I_X) = \text{rank}(T_N : O_N) = 1$.
- $\text{rank}(B_X : S_X) = \text{rank}(B_X : \text{Surj}_X) = \text{rank}(B_X : \text{Inj}_X) = 2$.

Further work: J.D. Mitchell, M. Morayne, Y. Peresse, J. Cichon, M. Quick, P. Higgins, NR.
Sierpinski rank

Definition
The Sierpinski rank of S is the smallest number \( n \) such that for any \( s_1, s_2, \ldots \in S \) there exist \( a_1, \ldots, a_n \in S \) such that \( s_i \in \langle a_1, \ldots, a_j \rangle \).

Proposition
\[
\text{srank}(S_X) = \text{srank}(T_X) = \text{srank}(I_X) = \text{srank}(P_X) = \text{srank}(B_X) = 2.
\]

Theorem (Peresse ’09)
\[
\text{srank}(\text{Inj}_X) = \begin{cases} 
  n + 4 & \text{if } |X| = \aleph_n \\
  \infty & \text{otherwise.}
\end{cases}
\]

Theorem (Mitchell, Peresse ’11)
\[
\text{srank}(\text{Surj}_X) = \begin{cases} 
  \frac{n^2 + 9n + 14}{2} & \text{if } |X| = \aleph_n \\
  \infty & \text{otherwise.}
\end{cases}
\]
Bergman property


**Definition**

$S$ has Bergman property if it has finite depth w.r.t. every generating set.

**Theorem (Bergman ’06)**

$S_X$ has Bergman property.

**Theorem**

$T_X, P_X, I_X, B_X$ all have Bergman property.

**Theorem**

*The finitary power semigroup of $I_X$ has Bergman property, but those of $T_X, P_X, B_X$ do not.*
$d$-sequences and relative rank

NR, Quick, Geddes, Hyde, Wallis, Loughlin, Carey, Awang, McLeman, Garrido.

\[ d_i = \text{rank}(S^i) \text{ (direct power)}. \]
\[ d(S) = (d_1, d_2, d_3, \ldots). \]

Relate asymptotic properties of $d(S)$ and algebraic properties of $S$.

\[ \Delta_n(S) = \{(s, \ldots, s) : s \in S\} \leq S^n. \]
\[ \overline{d}_i = \text{rank}(S^n : \Delta_n(S)). \]
\[ \overline{d}(S) = (\overline{d}_1, \overline{d}_2, \overline{d}_3, \ldots). \]

Proposition
\[ d(S) \sim \overline{d}(S). \]
Thank you