One Relation Semigroups

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Open Problem

Is the word problem soluble for every semigroup given by a single defining relation:

\[ \langle a_1, \ldots, a_n \mid u = v \rangle? \]
Presentations

$$\langle a_1, \ldots, a_n \mid u_1 = v_1, \ldots, u_m = v_m \rangle$$
The semigroup $S$ defined: the largest/free-est semigroup generated by (copies of) $a_1, \ldots, a_n$, in which these generators satisfy all relations $u_j = v_j$ (and their consequences, but nothing else).

How to think about $S$: elements are words over $\{a_1, \ldots, a_n\}$; some words are equal; two words are equal iff their equality is a consequence of the defining relations.

Example $S = \langle a, b | ba = a^2b \rangle$. Every word is equal to one of the form $a^i b^j$. 

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Nik Ruskuc: Residual Finiteness
Presentations

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letters/generators \hspace{1cm} words/defining relations

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**Example**

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Word Problem

Definition
A semigroup $S$ with a finite generating set $A$ has a **soluble word problem** if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of $S$. 

Example
$S = \langle a, b \mid ba = a^2b \rangle$. One can show: $a^i b^j = a^k b^l$ in $S$ $\iff i = k$ and $j = l$.

Algorithm for solving the word problem: Given two words $w_1, w_2$ transform them into $a^i b^j, a^k b^l$ and then test whether $i = k$ and $j = l$. 

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Brief Early History and Context

- 1900 – Hilbert’s 10th Problem: *Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*
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- 1912 – Dehn: formulation of the word problem for groups
- 1931 – Gödel: incompleteness theorems for 1st order theories
- 1932 – Magnus: word problem for one-relator groups
- 1947 – Markov, Post: finitely presented semigroups with insoluble word problems
- 1951 – Markov: undecidability galore
- 195? – Novikov, Britton, Boone: finitely presented groups with insoluble word problems
- 1979 – Matiyasevich: negative solution to Hilbert’s 10th Problem
Approaches

- Play with words (pages of induction 😕)
- Delegate (embeddings)
- Take apart (structure)
- Look at something else (other properties)
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Theorem (Magnus 1932)

*Every group defined by a single relation has a soluble word problem.*
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Theorem (Adyan 1966)

If $u$ and $v$ are non-empty words which have different first letters and different last letters then the semigroup defined by $\langle a_1, \ldots, a_n \mid u = v \rangle$ embeds into the group with the same presentation, and hence has a soluble word problem.
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Remark

Some descendants:

- Diagrams (Remmers 1971, 1980) and pictures (Pride 1993)
- Small overlap semigroups (Remmers)
- Applications: Kashintsev, Guba, Howie, Pride, Jackson,...
Other Types of Semigroups

Theorem (Adjan, Oganessian 1987)

One relation problem can be reduced to presentations of the type:

\[
\langle a, b \mid aua = avb, \rangle, \quad \langle a, b \mid a = avb \rangle
\]

Corollary

If every one relation right cancellative semigroup has a soluble word problem then every one relation semigroup has a soluble word problem.

Corollary (Ivanov, Margolis, Meakin 2001)

If every one relation inverse semigroup has a soluble word problem then every one relation semigroup has a soluble word problem.
Other Types of Semigroups

Theorem (Silva 1993)

One relation Clifford semigroups have a soluble word problem.

Question
How about completely regular semigroups?
Syntactical Approach: Special Monoids

Theorem (Adjan 1966)

Let $S$ be the monoid defined by

$$\langle a_1, \ldots, a_n \mid u = 1 \rangle.$$  

The group of units is one relator (but not necessarily same presentation). The semigroup $S$ has a soluble word problem.

Remark

See Zhang (1992) for a short proof and generalisation.
Magnus’s treatment of one relator groups: Freiheitssatz, ‘large’ subgroup, decompose into a product of free and/or ‘smaller’ one-relator groups.

Theorem (Semigroup Freiheitssatz; Squier, Wrathall 1983)

Let $S = \langle a_1, \ldots, a_n \mid u = v \rangle$ be a one relation semigroup, and suppose that $a_1$ appears in $u$ or $v$. Then the subsemigroup of $S$ generated by $\{a_2, \ldots, a_n\}$ is free.

Problem

- Investigate ‘large’ subsemigroups of one relation monoids.
- Candidates for large: $S \setminus \{a_1\}; \; S \setminus \langle a_1 \rangle; \ldots$
- Is there a natural decomposition?
- Do Rees index (Ruskuc 1998) or Green index (Gray, Ruskuc, to appear) help?
Other Properties

Investigate other structural, algebraic, combinatorial properties of one relation semigroups.

- Lallement 1974 – residual finiteness, idempotents
- Oganessian 1984 – isomorphism problem

A recent article:

What if it isn’t true?

Theorem (Matiyasevich 1967)
There exists a semigroup with three defining relations which has an insoluble word problem.

Theorem (Ivanov, Margolis, Meakin 2001)
Let $S$ be the inverse monoid defined by $\langle A \mid u = 1 \rangle$, where $w$ is a cyclically reduced word over $A \cup A^{-1}$. Let $G$ be the group defined by the same presentation, and let $P$ be the submonoid of $G$ generated by all the prefixes of $u$. Then $S$ has a soluble word problem if and only if the membership problem for $P$ is soluble.
P.S. Acknowledgement

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