Decidability Questions for Pattern Avoidance Classes of Permutations

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Template Question

Does there exist an algorithm which does the following:

**INPUT:** A (finite) (def)
defining a pattern avoidance class $X$ of permutations.

**OUTPUT:** TRUE if $X$ is (property), FALSE otherwise?

If such an algorithm exists we say that the property is decidable; if not, it is undecidable.
Algorithms

Algorithm

= Turing machine
= effective computation
= computer program
= unambiguous recipe

• know one when you see it;
• approach borrowed from algebra;
• a fresh look at the area;
• ignore technical issues: complexity, efficiency, etc (for the moment);
• computational tools would be useful nevertheless.
Permutations as Patterns

Sequence $=$ a finite list of distinct numbers.

Permutation $=$ a sequence of numbers $1, \ldots, n$ of length $n$.

Order isomorphism: for sequences $\sigma = s_1 \ldots s_m$, $\tau = t_1 \ldots t_n$ we say

$\sigma \cong \tau \iff m = n \& (\forall i, j : s_i \leq s_j \iff t_i \leq t_j)$.

Example. $8463 \cong 4231$, $8463 \not\cong 4321$.

Fact. For every sequence $\sigma$ there is a unique permutation $\bar{\sigma}$ such that $\sigma \cong \bar{\sigma}$.

Example. $\overline{8463} = 4231$. 
Involvement and Avoidance

**Definition.** \( \sigma = s_1 \ldots s_m \) is involved in \( \tau = t_1 \ldots t_n \) if \( \tau \) contains a subsequence order isomorphic to \( \sigma \):

\[
\sigma \preceq \tau \iff \exists 1 \leq i_1 < i_2 < \ldots < i_m \leq n : t_{i_1} \ldots t_{i_m} \simeq \sigma.
\]

If \( \sigma \not\preceq \tau \) we say that \( \tau \) avoids \( \sigma \).

**Example.** \( 123 \preceq 32415, 123 \not\preceq 35421 \).

**Fact.** \( \preceq \) is a preorder (RT) on the set \( T \) of all sequences, and is a (partial) order (RAST) on the set \( S \) of all permutations.

**Fact.** \( (T, \preceq) /\{(\sigma, \tau) : \sigma \preceq \tau \& \tau \preceq \sigma\} \cong (S, \preceq) \).
Closed (\(\equiv\) Pattern Avoidance) Classes

**Definition.** A set \(X \subset S\) of permutations is said to be closed if

\[
\sigma \in X \land \tau \preceq \sigma \Rightarrow \tau \in X.
\]

Another name: pattern avoidance class (why: see later).

**Examples.**

- \(S\);
- \(I = \{1, 12, 123, \ldots\}\);
- \(R = \{1, 21, 321, \ldots\}\);
- if \(X\) and \(Y\) are closed then so is \(X \cup Y\).
Defining Pattern Classes

- basis;
- token passing networks (TPNs);
- constructions:
  - union;
  - direct sum;
  - juxtaposition;
  - merge;
  - composition;
  - ...
- Sub() operator.
Properties

Definition/representation:

- finitely based;
- generated by a TPN;
- defined by Sub().

Enumeration:

- (a nice) recurrence;
- rational/algebraic/D-finite generating function;
- growth rate.

Structure:

- finite
- sum complete;
- atomic (union indecomposable);
- partially well ordered;
- ...
**Basis**

**Definition.** The basis $\mathcal{B}(X)$ of a closed class $X$ is the set of minimal elements not in $X$:

$$\mathcal{B}(X) = \{ \sigma \not\in X : (\forall \tau)(\tau \prec \sigma \Rightarrow \tau \in X) \}.$$ 

**Example.** $\mathcal{B}(S) = \emptyset$; $\mathcal{B}(I) = \{21\}$; $\mathcal{B}(I \cup R) = \{132, 213, 231, 312\}$.

**Definition.** For a set $Z \subseteq S$ its avoidance set is

$$\mathcal{A}(Z) = \{ \sigma : (\forall \tau \in Z)(\tau \not\in \sigma) \}.$$ 

**Facts.**

- $\mathcal{B}(X)$ is an antichain.
- $\mathcal{A}(Z)$ is closed.
- $\mathcal{A}(\mathcal{B}(X)) = X$. 
Basis $\rightarrow$ Finite

**Theorem.** A closed class $X$ is finite iff its basis contains $12\ldots m$ and $n\ldots 21$ for some $m$ and $n$. (And hence finiteness is decidable from the basis.)

**Proof.** ($\Leftarrow$) Erdös, Szekeres.

($\Rightarrow$) $X$ finite $\Rightarrow \exists m : 12\ldots m \notin X$. Choose the smallest such $m$. Then $12\ldots m \in \mathcal{B}(X)$. 
Direct Sum

**Definition.** For \( \sigma = s_1 \ldots s_m \) and \( \tau = t_1 \ldots t_n \) define

\[
\sigma \oplus \tau = s_1, \ldots, s_m, t_1 + m, \ldots, t_n + m.
\]

**Example.** \( 132 \oplus 213 = 132546 \).

**Definition.** \( X \oplus Y = \{ \sigma \oplus \tau : \sigma \in X, \tau \in Y \} \).

**Fact.** \( X, Y \) closed \( \Rightarrow \) \( X \oplus Y \) closed.

**Examples.** \( I \oplus I = I; \ R \oplus R = A(123, 312, 231) \).

**Definition.** \( X \) is sum complete if \( \sigma, \tau \in X \Rightarrow \sigma \oplus \tau \in X \).

**Examples.** \( I \) is sum complete, \( R \) is not.
Definition. \( \sigma \) is \textit{sum indecomposable} if \( \sigma \neq \tau \oplus \pi \) for any (non-empty) \( \tau, \pi \).

Theorem. A closed class \( X \) is sum complete iff every element of its basis is sum indecomposable.

Proof. (\( \Rightarrow \)) Suppose \( \sigma = \tau \oplus \pi \in \mathcal{B}(X) \). Then \( \tau, \pi \prec \sigma \) implies \( \tau, \pi \in X \), but \( \tau \oplus \pi \notin X \).

(\( \Leftarrow \))

\[
\begin{align*}
\sigma, \tau & \in X \\
\iff & \forall \beta \in \mathcal{B}(X) : \beta \not\preceq \sigma \& \beta \not\preceq \tau \\
\implies & \forall \beta \in \mathcal{B}(X) : \beta \not\preceq \sigma \oplus \tau \\
\iff & \sigma \oplus \tau \in X.
\end{align*}
\]
Sub() Operator

Definition. Let $A, B$ be (usually infinite) linearly ordered sets, and let $\pi : A \rightarrow B$ be a bijection. For every finite $C \subseteq A$, $\pi|_C$ is order isomorphic to a permutation. The set of all such permutations is denoted by $\text{Sub}(\pi : A \rightarrow B)$ or just $\text{Sub}(\pi)$.

Fact. $\text{Sub}(\pi)$ is closed.

Example. $\pi : \mathbb{N} \rightarrow -\mathbb{N}$, $\pi(x) = -x$. $\text{Sub}(\pi) = R$.

Example. $\pi : \mathbb{N} \rightarrow \mathbb{N}$, $1 \leftrightarrow 2$, $2 \leftrightarrow 1$, $3 \leftrightarrow 3$, $4 \leftrightarrow 4$, ... $\text{Sub}(\pi) = \{12...n : n \geq 1\} \cup \{2134...n : n \geq 2\}$. 
Unions, Sub and Join Property

**Theorem.** [Atkinson, Murphy, NR] The following conditions are equivalent for a closed class $X$:

(i) $X \neq Y \cup Z$ for any proper subclasses $Y, Z$.

(ii) $X = \text{Sub}(\pi : A \to B)$ for some $\pi, A, B$.

(iii) $X$ has the join property:

\[
\forall \sigma, \tau \in X : \exists \pi \in X : \sigma \preceq \pi \text{ & } \tau \preceq \pi.
\]

**Definition.** A class satisfying one (and hence all) of the above conditions is said to be atomic.
Example. \( I, R, \text{Sub}(2134\ldots) \) and \( R \oplus R \) are atomic. \( I \cup R \) is not atomic.

Fact. Sum complete \( \Rightarrow \) atomic.

Open Question. Is atomicity decidable from the basis?
Sub(\(\pi : \mathbb{N} \rightarrow \mathbb{N}\)) (Natural Classes)

**Fact.** Sum complete \(\Rightarrow\) natural.

**Fact.** \(\gamma \in S\), \(X\) sum complete \(\Rightarrow\) Sub(\(\gamma\)) \(\oplus\) \(X\) natural.

**Theorem.** [Atkinson, Murphy, NR] If \(X\) is a finitely based natural class then one of the following holds:

(i) \(X = \text{Sub}(\gamma) \oplus Y\), where \(\gamma \in S\), \(Y\) is sum complete and uniquely determined by \(X\); or

(ii) \(X = \text{Sub}(\pi : \mathbb{N} \rightarrow \mathbb{N})\) where \(\pi\) is uniquely determined by \(X\) and is ultimately periodic:

\[\exists N, \omega : \forall n \geq N : \pi(n + \omega) = \pi(n) + \omega.\]
Corollary. [Murphy] It is decidable whether a finitely based closed class is natural.

Open Questions. Is it decidable whether a finitely based class can be written as any of the following:

- \text{Sub}(\pi: \mathbb{Z} \rightarrow \mathbb{Z})?
- \text{Sub}(\pi: 2\mathbb{N} \rightarrow \mathbb{N})?
- \text{Sub}(\pi: \mathbb{Q} \rightarrow \mathbb{Q})?
**Enumeration**

\[ s_n(X) = \text{the number of permutations of length } n \text{ in } X. \]

**Example.** \[ s_n(I) = s_n(R) = 1; \ s_n(R \oplus R) = n. \]

**Theorem.** [Knuth; Simion, Schmidt] If \( \pi \) is a permutation of length 3 then \( s_n(A(\pi)) = C_n \), the \( n \)th Catalan number.

**Remarks.** Exact formulae are known for \( s_n(A(\pi)) \) where \( \pi \) is any permutation of length 4, except when \( \pi \) is (equivalent to) 1324. Very little is known about \( s_n(A(\pi)) \) when \( |\pi| \geq 5 \).
Enumeration:
Generating Functions

Open Questions. Is it decidable whether the ordinary generating function of $s_n(X)$ for a finitely based closed class $X$ is (a) rational? (b) algebraic? (c) $D$-finite?

Theorem. [Bousquet-Melou] The generating function for $\mathcal{A}(1234)$ is $D$-finite but not algebraic.

Question. [Gessel] Is the generating function for a finitely based closed class always $D$-finite?

Theorem. [Murphy] There exists an infinitely based closed class the generating function of which is not $D$-finite.
Enumeration: Growth

**Theorem.** If $X \neq S$ is a closed class then there exists $q$ such that $s_n(X) \leq q^n$.

**Remarks.** Proved very recently by Marcus and Tardos. Before that it was known as the Wilf–Stanley Conjecture.

**Corollary.** The limit $q = \lim_{n \to \infty} n^{\sqrt{s_n(X)}}$ exists. (The growth of $X$.)

**Examples.** The growth of $\mathcal{A}(\pi)$ where $|\pi| = 3$ is 4. [Regev] The growth of $\mathcal{A}(12\ldots k)$ is $(k - 1)^2$.

**Questions.** For a fixed $q$, is it decidable whether $q$ is the growth of a finitely based class $X$? Is the growth of a finitely based class effectively computable?

**Remark.** [Bona] The growth of $\mathcal{A}(12453)$ is $9 + 4\sqrt{2}$. 
Token Passing Networks (TPNs)

A TPN is a finite directed graph with a distinguished input vertex I and a distinguished output vertex O. Each vertex is one of the following:

- simple node, capable of holding one item of data;
- stack, capable of holding any number of items, and treating them in the FILO discipline;
- queue (FIFO);
- . . .

If all vertices are of type (i) we have a finite capacity (FC) TPN.
TPNs

Let $N$ be a TPN. $N$ can generate permutations: feed $12 \ldots n$ into $N$, item by item, through $I$, move items along edges respecting the orientation, store them in vertices respecting the type, output them via $O$.

$\mathcal{P}(N) =$ the set of all output permutations.

**Facts.** $\mathcal{P}(N)$ is closed, sum complete and atomic.

**Remark.** Now atomicity is decidable!
TPN: Example

\[ X = \mathcal{P}(N) \]

- \( X \) is not finitely based.
- Unknown: \( \mathcal{B}(X); s_n(X); \) growth; enumeration for the basis.
FCTPNs

Let $N$ be a FCTPN, $|N| = m$, $X = \mathcal{P}(N)$.

**Theorems.** (i) [Atkinson, Livsey, Tulley] Elements of $N$ can be encoded, symbol by symbol, by words over an alphabet of size at most $m$. [The actual size of the alphabet will be called the boundedness of $X$.] The resulting set of words is a regular language.

(ii) The generating function for $X$ is rational and can be effectively computed.

(iii) [Albert, Atkinson, NR] $\mathcal{B}(X)$ can be encoded in the same fashion, over the same alphabet, again yielding a regular language.

(iv) It is decidable whether $X$ is finitely based.

**Remark.** The above algorithms are practical and have been implemented in GAP.
**FCTPNs**

**Theorem.** [Albert, Linton, NR] For any $m$ there are only finitely many pattern classes of the form $\mathcal{P}(N)$, where $N$ is a FCTPN of boundedness $m$.

**Fact.** If a class $X$ is given by its basis, it is decidable whether $X$ is bounded.

**Question.** Is it decidable whether a given finitely based classs is generated by a FCTPN?
TPNs

Question. Is it decidable whether a given TPN generates a finitely based class?

Theorem. [Waton] It is decidable whether a given TPN generates $S$, the class of all permutations.

Idea. Avoid
Partial Well Order:
Towards Undecidability?

**Definition.** [Following G. Higman] A pattern class $X$ is said to be partially well ordered (PWO) if it contains no infinite antichain.

**Theorem.** [Atkinson, Murphy, NR] A finitely based closed class is PWO iff it has only countably many subclasses.

**Examples.**

- $\mathcal{A}(12)$ is PWO.
- [Atkinson, Murphy, NR] $\mathcal{A}(231)$ is PWO.
- [Spielman, Bona] $\mathcal{A}(123)$ is not PWO.

**Questions.** Is partial well orderedness decidable for closed classes given by (a) finite bases? (b) TPNs? (c) FCTPNs?
> f:=proc(list)
> RETURN(true);
> end;
>

    f := proc(list) RETURN(true) end proc

> f([2,3,1],[3,5,2,1,4]);
true