Generators and Defining Relations for Direct Products of Semigroups

Nik Ruškuc
St Andrews, Scotland, UK
www-groups.mcs.st-and.ac.uk/∼nik

With:

Isabel M Araujo (Evora)
Edmund F Robertson (St And.)
Jim Wiegold (Cardiff)

NSAC 2003
Novi Sad, August 2003

Project: Presentations for Semigroup Constructions

Construction: Take semigroups S_i $(i \in I)$ and form a new semigroup S, according to a 'recipe'.

Question. How are presentations for S_i $(i \in I)$ related to presentations for S?

Examples.

- Products: direct, semidirect, wreath,. . . ;
- Rees matrix semigroups;
- Bruck–Reilly extensions;
- strong semilattices of semigroups;
- etc. etc.

Presentations (1)

- A – an alphabet.
- A^+ – non-empty words over A; free semigroup.
- 1 – the empty word.
- $A^* = A \cup \{1\}$ – the free monoid.
- $R \subseteq A^+ \times A^+$ – defining relations; we write $u = v$ instead of (u, v).
- $\langle A \mid R \rangle$ – a (semigroup) presentation.

Presentations (2)

- $\langle A \mid R \rangle$ – a presentation.
- ρ – the smallest congruence on A^+ containing R.
- $S \cong A^+/\rho$ – the semigroup defined by $\langle A|R \rangle$.

Definition. A semigroup S is finitely presented if it can be defined by $\langle A \mid R \rangle$ with both A and R finite.

Examples.

- A^+ (with A finite) is finitely presented.
- Every finite semigroup is f.p.
- Free monoids and groups of finite rank are f.p.
- $\langle a, b \mid ab^i a = aba \ (i = 2, 3, \ldots) \rangle$ is not f.p.
- Free inverse semigroups are not f.p.

Presentations (3)

- $\langle A \mid R \rangle$ – a presentation.
- $S = A^+/\rho$ – the semigroup defined by $\langle A|R \rangle$.
- $\alpha, \beta \in A^+$.
- $\alpha \to \beta$ – if $\alpha \equiv \gamma u \delta$, $\beta \equiv \gamma v \delta$, where $(u, v) \in R$ or $(v, u) \in R$.

Fact. Two words w_1, w_2 represent the same element of S (i.e. $w_1/\rho = w_2/\rho$) iff

$$w_1 \equiv \alpha_1 \to \alpha_2 \to \ldots \to \alpha_n \equiv w_2$$

for some $n \geq 1$, $\alpha_i \in A^+$.

Monoid Presentations

The only difference between the monoid presentations and semigroup presentations is that in the monoid presentations we allow relations of the form $u = 1$.

Example. The bicyclic monoid $\langle b, c \mid bc = 1 \rangle$.

Fact. A monoid M is finitely presented as a monoid if and only if it is finitely presented as a semigroup.

Two Questions

Question 1. Under what conditions is the direct product $S \times T$ of two semigroups finitely generated?

Question 2. Under what conditions is the direct product $S \times T$ of two semigroups finitely presented?

Monoids

> **Theorem.** Let M and N be two monoids. Then the direct product $M \times N$ is finitely presented if and only if both M and N are finitely presented.

Proof. ($\Leftarrow$) If $M = \langle A \,|\, R \rangle$ and $N = \langle B \,|\, Q \rangle$ then

$$M \times N = \langle A, B \,|\, R,\, Q,\, ab = ba\,(a \in A,\, b \in B) \rangle$$

in terms of the generating set $\{(a, 1), (1, b) : a \in A, b \in B\}$.

($\Rightarrow$) Project any finite presentation for $M \times N$ in terms of a 'natural' generating set onto the factors.

Convention

From now on, S and T denote two arbitrary infinite semigroups.

Finite generation (1)

Definition. An element $s \in S$ is said to be indecomposable if s is not a product of two elements from S, i.e. if $s \notin S^2$.

Example. The element 1 is indecomposable in the additive semigroup $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers.

Facts.

- An indecomposable element must belong to every generating set of S.
- S has no indecomposable elements if and only if $S^2 = S$.
- If $s \in S$ is indecomposable then (s, t) is indecomposable in $S \times T$ for every $t \in T$.
- If $S^2 \neq S$ then $S \times T$ is not finitely generated.

Finite generation (2)

Facts.

- Every (finite) generating set for a semigroup S satisfying $S^2 = S$ can be extended to a (finite) generating set A for S satisfying $A \subseteq A^2$.
- In terms of such a generating set, if an element of S can be represented by a word of length m, then it can also be represented by a word of length n for any $n \geq m$.

Finite generation (3)

- Assume now that S and T are finitely generated and that $S^2 = S$ and $T^2 = T$.
- Let $S = \langle A \rangle$, $|A| < \infty$, $A \subseteq A^2$.
- Let $T = \langle B \rangle$, $|B| < \infty$, $B \subseteq B^2$.
- Let $s \in S$ and $t \in T$ be arbitrary.
- Represent s and t by two words $a_1a_2 \ldots a_n$ and $b_1b_2 \ldots b_n$ of equal lengths over A and B respectively.
- Then $(s, t) = (a_1, b_1)(a_2, b_2) \ldots (a_n, b_n)$.
- Hence $S \times T = \langle A \times B \rangle$.

Finite generation (4)

Theorem. Let S and T be two infinite semigroups. Then $S \times T$ is finitely generated if and only if both S and T are finitely generated and satisfy $S^2 = S$ and $T^2 = T$.

Finite presentability (1)

- Assume now that $S \times T$ is finitely presented.
- Let $S = \langle A \rangle$, $T = \langle B \rangle$ as before.
- We know $S \times T = \langle A \times B \rangle$.
- Let $\langle A \times B \,|\, R \rangle$ be any finite presentation for $S \times T$.
- $\pi_A : (A \times B)^+ \to A^+$, $(a, b) \mapsto a$.
- $\pi_B : (A \times B)^+ \to B^+$, $(a, b) \mapsto b$.
- π_A and π_B preserve length, i.e. $|w| = |\pi_A(w)|$ $= |\pi_B(w)|$ for every $w \in (A \times B)^+$.
- Let $u, v \in A^+$ be any two words such that $u = v$ in S.
- Assume, in addition, that $|u| = |v|$ and write $u \equiv a_1 a_2 \dots a_n$, $v \equiv a'_1 a'_2 \dots a'_n$. (This assumption is not essential.)
- Let $w \in B^+$ be a word of length n, which is not equal in T to a shorter word (T is infinite!); write $w \equiv b_1 b_2 \dots b_n$.

Finite presentability (2)

- Let $\alpha \equiv (a_1, b_1)(a_2, b_2) \ldots (a_n, b_n)$ and $\beta \equiv (a'_1, b_1)(a'_2, b_2) \ldots (a'_n, b_n)$.
- $\alpha = \beta$ holds in $S \times T$.
- So: $\beta \equiv \gamma_1 \to \gamma_2 \to \ldots \to \gamma_k \equiv \beta$.
- Then: $u \equiv \pi_A(\alpha) \equiv \pi_A(\gamma_1) \to \pi_A(\gamma_2) \to \ldots \to \pi_A(\gamma_k) \equiv \pi_A(\beta) \equiv v$.
- Hence $S = \langle A \,|\, \pi_A(R) \rangle$, and so both S and T must be finitely presented.
- Second components: $w \equiv \pi_B(\alpha) \equiv \pi_B(\gamma_1) = \pi_B(\gamma_2) = \ldots = \pi_B(\gamma_k) \equiv \pi_B(\beta) \equiv w$.
- By the choice of w: $|\pi_B(\gamma_i)| \geq n$ $(1 \leq i \leq n)$.
- Hence $|\pi_A(\gamma_i)| \geq n$ $(1 \leq i \leq n)$.

Critical pairs and stability

Definition. Let $\langle A \,|\, R \rangle$ be a presentation, let S be the semigroup defined by it, and let $w_1, w_2 \in A^+$ be arbitrary. The pair (w_1, w_2) is called a critical pair if

(i) $w_1 = w_2$ in S; and

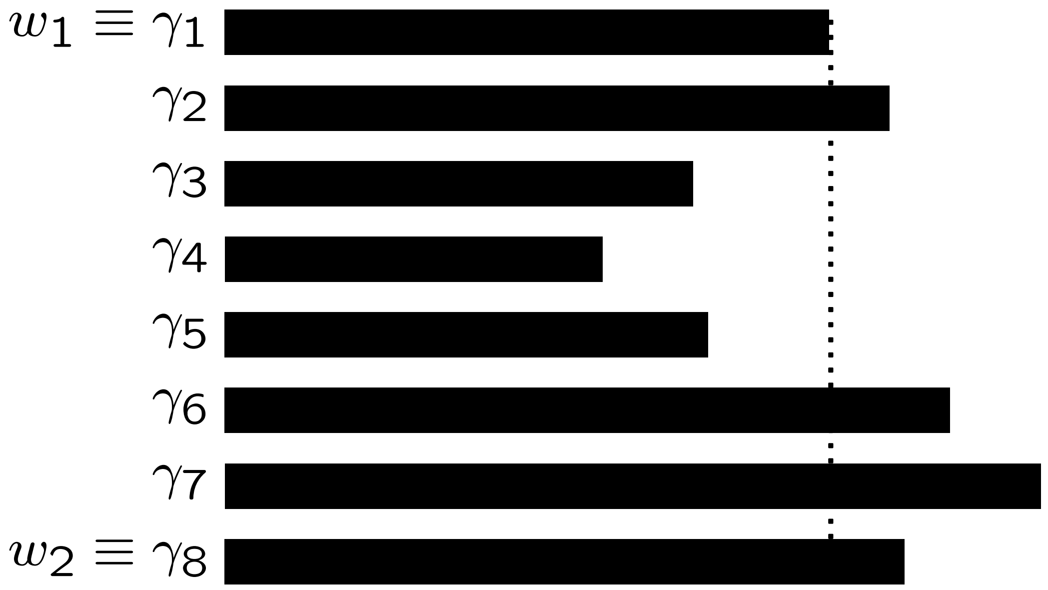
(ii) for every sequence $w_1 \equiv \alpha_1 \to \alpha_2 \to \ldots \to \alpha_k \equiv w_2$ there is i such that $|\alpha_i| < \min(|w_1|, |w_2|)$.

Definition. A semigroup S is said to be stable if it can be defined by a finite presentation $\langle A \,|\, R \rangle$ with respect to which it has no critical pairs.

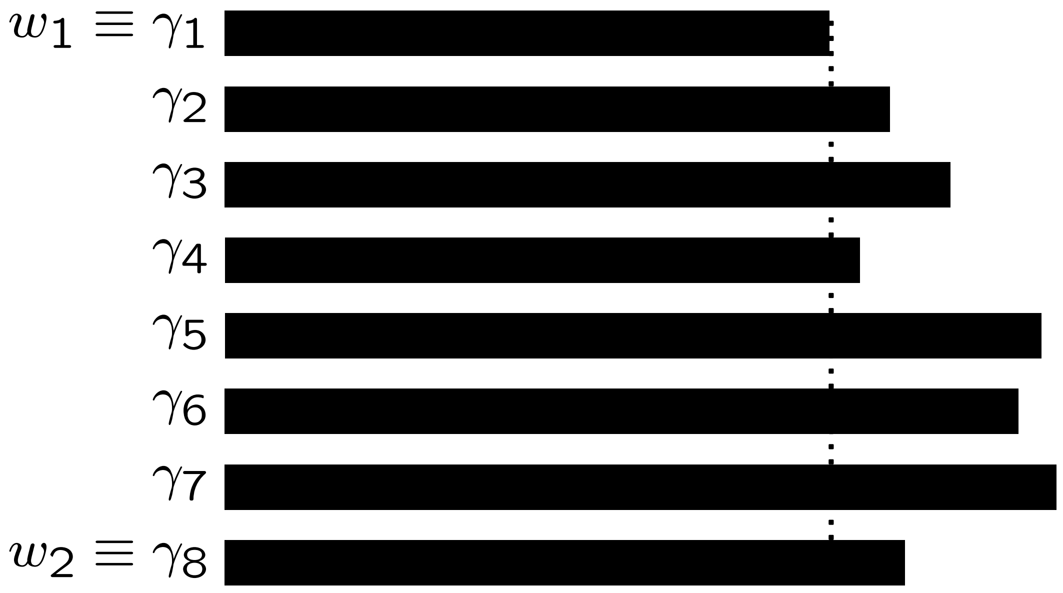
Fact. Non-existence of critical pairs is invariant under change of generators, but is not invariant under change of presentation.

Critical pairs

Critical:



Non-critical:



Finite presentability (3)

Theorem. (Robertson, NR and Wiegold, 1998) Let S and T be two infinite semigroups. Then $S \times T$ is finitely presented if and only if

(i) $S^2 = S$ and $T^2 = T$; and

(ii) S and T are (finitely presented and) stable.

Stability and identity elements (1)

Theorem. Either of following conditions is sufficient for stability of a semigroup S:

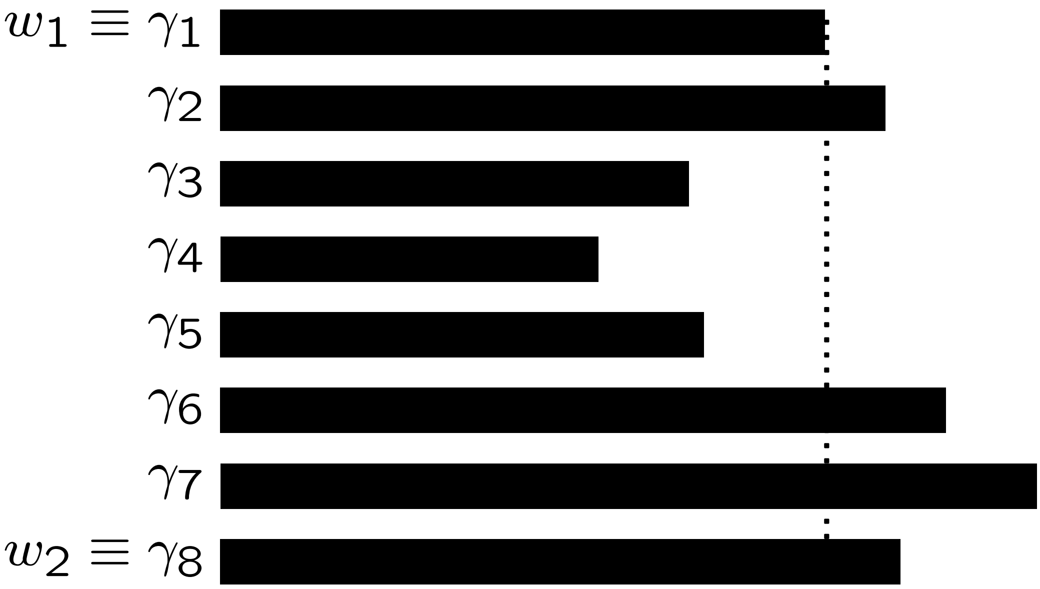
(i) S has a one sided identity.

(ii) Every element of S has its own left identity and its own right identity, i.e. $(\forall s \in S)(\exists e_s, f_s \in S)(e_s s = s f_s = s)$.

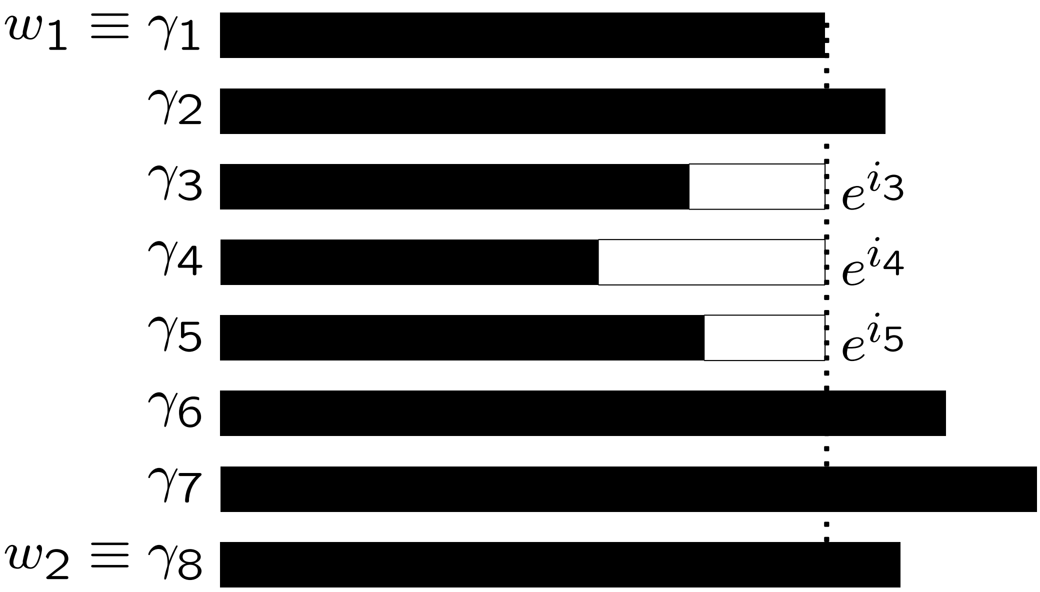
Remark. We have (i)$\nRightarrow$(ii) and (ii)$\nRightarrow$(i). Hence neither (i) nor (ii) is a necessary condition for stability.

Stability and identity elements (2)

$S = \langle A \,|\, R \rangle$:



$S = \langle A, e \,|\, R,\ e = w,\ e^2 = e,\ ae = a\,(a \in A) \rangle$:



Direct products of special types of semigroups

Corollary. Let S and T belong (independently) to any of the following classes:

- monoids (including groups);
- regular semigroups $((\forall x \in S)(\exists y \in S)(xyx = x))$; this includes inverse semigroups, unions of groups, completely 0-simple semigroups, etc.;
- commutative semigroups satisfying $S^2 = S$.

Then $S \times T$ is finitely presented if and only if S and T are finitely presented.

A non-stable semigroup

- $S = \langle a, x, y \,|\, xa = a,\ ya = a,\ xy = x,\ y^2 = y \rangle$.
- $S^2 = S$; hence, for any T, $S \times T$ is finitely generated if and only if T is finitely generated and $T^2 = T$.
- For any finite presentation $\langle a, x, y \,|\, R \rangle$ there is m such that $(x^m a, y^m a)$ is a critical pair.
- For example, (xa, ya) is a critical pair w.r.t. the given presentation.
- Thus S is not stable.
- Hence $S \times T$ is never finitely presented.

A Problem

Problem. Is it decidable whether a finitely presented semigroup is stable?

$$\Updownarrow$$

Problem. Is it decidable whether the direct product of two finitely presented semigroups is finitely presented?

Remark. It is decidable whether $S^2 = S$. So it is decidable whether the direct product of two (finitely presented) semigroups is finitely generated.

Convention

From now on S will denote a finite semigroup and T will denote an infinite semigroup.

Finite $\times$ infinite (1)

Theorem. Let S be a finite semigroup and let T be an infinite semigroup. Then $S \times T$ is finitely generated if and only if $S^2 = S$ and T is finitely generated.

Theorem. Let S be a finite semigroup and let T be an infinite semigroup. Then $S \times T$ is finitely presented if and only if S is stable and satisfies $S^2 = S$, and T is finitely presented.

Finite $\times$ infinite (2)

Corollary. Let S be a finite semigroup. Then precisely one of the following statements is true.

(I) $S \times T$ is not finitely generated for any infinite semigroup T.

(II) $S \times T$ is finitely generated if and only if T is finitely generated, but it is not finitely presented for any infinite T.

(III) $S \times T$ is finitely generated (resp. presented) if and only if T is finitely generated (resp. presented).

Finite $\times$ infinite (3)

Remarks.

- S satisfies (I) if and only if $S^2 \neq S$.
- S satisfies (II) if and only if $S^2 = S$ and S is not stable.
- S satisfies (III) if and only if $S^2 = S$ and S is stable.

Question. Is there a constructive way of distinguishing between types (II) and (III)?

Finite $\times$ infinite (4)

- $R = \{s \in S : s$ generates a maximal principal right ideal$\}$.
- $L = \{s \in S : s$ generates a maximal principal left ideal$\}$.

Definition. For $s \in S$ define a graph $\Gamma(s)$ as follows.

- Vertices:

$$\{(\alpha, \mu, \omega) : \alpha \in R,\ \mu \in S,\ \omega \in L,\ \alpha\mu\omega = s\}.$$

- Edges: $(\alpha_1, \mu_1, \omega_1) \sim (\alpha_2, \mu_2, \omega_2)$ if and only if one of the following holds:
  (i) $\alpha_1 = \alpha_2$ and $\mu_1\omega_1 = \mu_2\omega_2$; or

  (ii) $\alpha_1\mu_1 = \alpha_2\mu_2$ and $\omega_1 = \omega_2$.

Finite $\times$ infinite (5)

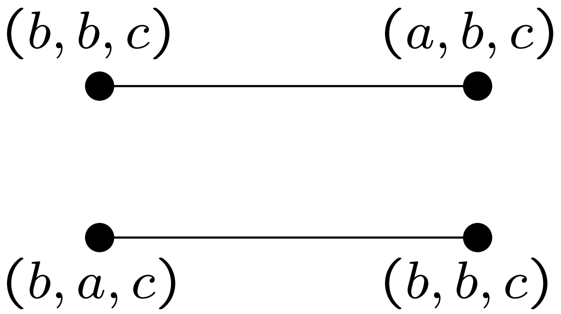
Theorem. (Araújo and NR, 2000) Let S be a finite semigroup. Then S preserves finite generation and presentability of direct products (i.e. S is of type (III)) if and only if all the graphs $\Gamma(s)$ ($s \in S$) are connected.

Example (1)

S	a	b	c	d
a	a	a	c	d
b	b	b	c	d
c	d	d	d	d
d	d	d	d	d

$R = \{a, b\}$,
$L = \{a, b, c\}$

$\Gamma(c)$:



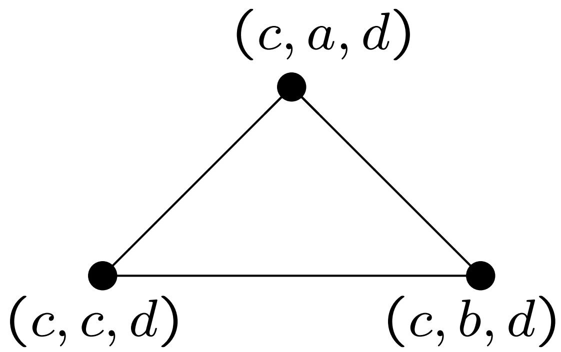
- S is of type (II) (i.e. it preserves finite generation but not finite presentability).
- Every semigroup of order ≤ 3 is either of type (I) or type (III).

Example (2)

S	a	b	c	d	e	f
a	a	a	a	d	f	f
b	b	b	b	d	f	f
c	a	b	c	d	f	f
d	f	f	f	f	f	f
e	f	f	f	f	e	f
f	f	f	f	f	f	f

$R = \{c, e\}$,
$L = \{c, d, e\}$

- $\Gamma(a) = \{(c, a, c)\}$, $\Gamma(b) = \{(c, b, c)\}$, $\Gamma(c) = \{(c, c, c)\}$, $\Gamma(e) = \{(e, e, e)\}$.
- $\Gamma(d)$:



- $\Gamma(f)$ has 29 vertices and is connected.
- S is of type (III), although S has no one-sided identity and the element d has no relative right identity.

References

[1] I. Araújo and N. Ruškuc, On finite presentability of direct products of semigroups, *Algebra Colloq.* **7** (2000), 83–91.

[2] E.F. Robertson, N. Ruškuc and J. Wiegold, Generators and relations of direct products of semigroups, *Trans. Amer. Math. Soc.* **350** (1998), 2665–2685.