Subsemigroups of the Bicyclic Monoid

Nik Ruškuc
Joint work with: Luis Descalço
St Andrews, Scotland, U.K.
nik@mcs.st-and.ac.uk
http://www-groups.mcs.st-and.ac.uk/~nik

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Bicyclic Monoid $B$

**Definition.**  
$B = \langle b, c \mid bc = 1 \rangle = \{c^i b^j : i, j = 0, 1, 2, \ldots \}$.

**Facts.**  

- $B$ is a bi-simple inverse monoid.
- $E(B) = \{c^i b^i \mid i = 0, 1, 2, \ldots \}$, and this is a subsemigroup (down-chain).
- For every $m = 0, 1, 2, \ldots$ each of the sets $\{c^i b^i : i > m\}$ and $\{c^i b^m : i > m\}$ is a subsemigroup isomorphic to $\mathbb{N}$ (additive).
- The mapping $c^i b^j \mapsto c^j b^i$ is an anti-isomorphism.
- The lattice of subsemigroups of $B$ is distributive (Jones 1978).
- $B$ is determined by its subsemigroup lattice (Shevrin 1993).
Sub(semi)groups of $\mathbb{Z}$ and $\mathbb{N}$

**Fact.** Every subgroup of $\mathbb{Z}$ is cyclic, and of the form

$$d\mathbb{Z} = \{da : a \in \mathbb{Z}\}.$$

**Fact.** Every subsemigroup of $\mathbb{N}$ has the form

$$F \cup D_{C,d},$$

where $F$ is finite and

$$D_{C,d} = \{da : a \geq C\}$$

for some $C$. (Sylvester’s stamps:-)
Main Theorem

**Theorem.** Every subsemigroup of $B$ is of one of the following five types:

1 **Diagonal:** a subset of $E(B)$.

2 **Upper two-sided:** $F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$.
   (Here $F_D \subseteq E(B)$, $F \subseteq B$, $I, P \subseteq \mathbb{N}$ are finite sets, while $p$ and $d$ are integers.)

3 **Lower two-sided:** dual to 2.

4 **Upper one-sided:** $F_D \cup \bigcup_{i \in I}(F_i \cup \Lambda_{i,m_i,d})$, where $I = I_0 \cup \bigcup_{r \in R}(r + D_{C,d})$.
   (Here $F_D \subseteq E(B)$, $F_i \subseteq B$, $I_0, R \subseteq \mathbb{N}$ are finite sets, while $d$, $C$ and $m_i$ are integers.)

5 **Lower one-sided:** dual to 5.

Moreover, all the above parameters can be effectively computed from a generating set.
Corollaries

**Corollary.** A subsemigroup of $B$ is finitely generated if and only if it is one of the following:

- finite diagonal; or
- two-sided; or
- one-sided with finitely many rows or columns.

**Corollary.** Every finitely generated subsemigroup of $B$ is finitely presented.

**Corollary.** Every finitely generated subsemigroup of $B$ is automatic.

**Corollary.** A subsemigroup of $B$ is residually finite if and only if it is diagonal or one-sided.