

Subsemigroups of the Bicyclic Monoid

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Pisa, Italy, June 2002

Bicyclic Monoid B

Definition. $B = \langle b, c \mid bc = 1 \rangle$
 $= \{c^i b^j : i, j = 0, 1, 2, \dots\}.$

Facts.

- B is a bi-simple inverse monoid.
- $E(B) = \{c^i b^i \mid i = 0, 1, 2, \dots\}$, and this is a subsemigroup (down-chain).
- For every $m = 0, 1, 2, \dots$ each of the sets $\{c^m b^i : i > m\}$ and $\{c^i b^m : i > m\}$ is a subsemigroup isomorphic to \mathbb{N} (additive).
- The mapping $c^i b^j \mapsto c^j b^i$ is an anti-isomorphism.
- The lattice of subsemigroups of B is distributive (Jones 1978).
- B is determined by its subsemigroup lattice (Shevrin 1993).

Sub(semi)groups of \mathbb{Z} and \mathbb{N}

Fact. Every subgroup of \mathbb{Z} is cyclic, and of the form

$$d\mathbb{Z} = \{da : a \in \mathbb{Z}\}.$$

Fact. Every subsemigroup of \mathbb{N} has the form

$$F \cup D_{C,d},$$

where F is finite and

$$D_{C,d} = \{da : a \geq C\}$$

for some C . (Sylvester's stamps:-)

Main Theorem

Theorem. Every subsemigroup of B is of one of the following five types:

1 Diagonal: a subset of $E(B)$.

2 Upper two-sided: $F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$.
(Here $F_D \subseteq E(B)$, $F \subseteq B$, $I, P \subseteq \mathbb{N}$ are finite sets, while p and d are integers.)

3 Lower two-sided: dual to 2.

4 Upper one-sided: $F_D \cup \bigcup_{i \in I} (F_i \cup \Lambda_{i,m_i,d})$,
where $I = I_0 \cup \bigcup_{r \in R} (r + D_{C,d})$. (Here $F_D \subseteq E(B)$, $F_i \subseteq B$, $I_0, R \subseteq \mathbb{N}$ are finite sets, while d, C and m_i are integers.)

5 Lower one-sided: dual to 4.

Moreover, all the above parameters can be effectively computed from a generating set.

Corollaries

Corollary. A subsemigroup of B is finitely generated if and only if it is one of the following:

- finite diagonal; or
- two-sided; or
- one-sided with finitely many rows or columns.

Corollary. Every finitely generated subsemigroup of B is finitely presented.

Corollary. Every finitely generated subsemigroup of B is automatic.

Corollary. A subsemigroup of B is residually finite if and only if it is diagonal or one-sided.