Rewriting Generators for (Sub)Semigroups

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Porto, 6 July 2009
Setting the Scene

$S$ – a semigroup/monoid

$S = \langle A \rangle$

$T \leq S$
Setting the Scene

$S$ – a semigroup/monoid

$S = \langle A \rangle$

$T \leq S$

Find a generating set for $T$. 
Setting the Scene

$S$ – a semigroup/monoid

$S = \langle A \rangle$

$T \leq S$

Find a ‘nice’ generating set for $T$. 

People

- Alan J. Cain, Robert Gray, NR, Green index in semigroups: generators and related finiteness conditions, about to be submitted.
- Also: C.M. Campbell, M. Hoffmann, E.F. Robertson, R.M Thomas.
- Related: R. Gray, V. Maltcev, J.D. Mitchell, NR, Ideals, finiteness conditions and Green index for subsemigroups, about to be submitted.
A first naive approach

Given a ‘long’ word from $A^*$ representing an element of $T$, ‘decompose’ it into ‘shorter’ members of $T$. 

Example $S = \{a, b\}$ $T = \{\text{words of even length}\}$

Hence $T = \langle a_2, ab, ba, b_2 \rangle$. 

Nik Ruskuc: Rewriting for (sub)semigroups
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\[ a_1a_2a_3 \ a_4a_5 \ a_6 \ a_7a_8a_9 \ldots \]

$\notin T \quad \in T$
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\end{align*}$

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& \notin T & \in T
\end{align*}$

Example

$S = \{ a, b \}^*$

$T = \{ \text{words of even length} \}$
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\]

\(! \in T\) \hspace{2cm} \in T

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$\overset{\not\in}a_1a_2a_3\mid a_4a_5\mid a_6\mid a_7a_8a_9\mid \ldots$

$\in T$

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$ab \mid aa \ ba \ bb$
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\[ ab|aa|ba|bb \]

Hence $T = \langle a^2, ab, ba, b^2 \rangle$. 
Variation: finite complement (Rees index)

Suppose $|S \setminus T| = r < \infty$

Theorem (Jura 1978)

If $S = \langle A \rangle$ then $T = \langle \{ uav : u, v \in (S \setminus T), a \in A, ua, uav \in T \} \rangle$.

Corollary

$S$ is finitely generated if and only if $T$ is finitely generated.
Variation: finite complement (Rees index)

Suppose $|S \setminus T| = r < \infty$

Consider a product $a_1a_2\ldots a_{r+1}$. Then:

(i) $\exists k$: $a_1\ldots a_k \in T$, or
(ii) $\exists k, l$: $a_1\ldots a_k = a_1\ldots a_l$.

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Rewriting mapping

Suppose $S = \langle A \rangle$, $T \leq S$, $T = \langle B \rangle$. Let $L(A, T) = \{\text{words from } A^* \text{ representing elements of } T\}$. Then there exists a mapping $\phi: L(A, T) \to B^*$ such that $w \in L(A, T)$ and $\phi(w) \in B^*$ represent the same element of $T$.

From the previous slides: For $w \in L(A, T)$ let $w = uav$, where $u$ is the shortest prefix in $L(A, T)$. Then $\phi(w) = \begin{cases} (u \cdot \phi(v)) & \text{if } v \in L(A, T) \\ w & \text{otherwise} \end{cases}$. 

Nik Ruskuc: Rewriting for (sub)semigroups
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$\mathcal{L}(A, T) = \{\text{words from } A^* \text{ representing elements of } T\}$

Then there exists a mapping

$$\phi : \mathcal{L}(A, T) \to B^*$$

such that $w \in \mathcal{L}(A, T)$ and $\phi(w) \in B^*$ represent the same element of $T$. 
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Another idea: ‘Coset representatives’
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\[
\{ h_i : i \in I \} \\
\tau(i, s) = \tau(i, s) \in T, \lambda(i, s) \in I
\]
Another idea: ‘Coset representatives’

\[ \text{\{h}_i : i \in I \}\]  

\[ h_i = \tau(i, s) = \lambda(i, s) \]  

\[ \tau(i, s) \in T, \lambda(i, s) \in I \]  

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\[ \tau(i, s) \in T, \ \lambda(i, s) \in I \]
\[ h_is = \tau(i, s)h_{\lambda(i,s)} \]
\[ h_i s = \tau(i, s) h_{\lambda(i, s)} \]

\[ a_1 a_2 a_3 \ldots a_k \]
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\[ 1 \ a_1 a_2 a_3 \ldots a_k \]
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\[ h_{i_1} a_1 a_2 a_3 \ldots a_k \]
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\[ h_{i_1}a_1a_2a_3 \ldots a_k \]

\[ = \tau(i_1, a_1)h_\lambda(i_1, a_1)a_2a_3 \ldots a_k \]
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\[ h_{i_1} a_1 a_2 a_3 \ldots a_k \]

\[ = \tau(i_1, a_1) h_{i_2} a_2 a_3 \ldots a_k \]
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\[
\begin{align*}
  h_{i_1} a_1 a_2 a_3 \ldots a_k &= \tau(i_1, a_1) h_{i_2} a_2 a_3 \ldots a_k \\
  &= \ldots
\end{align*}
\]
\[ h_is = \tau(i, s)h_{\lambda(i,s)} \]

\[
\begin{align*}
 & h_{i_1} a_1 a_2 a_3 \ldots a_k \\
 = & \quad \tau(i_1, a_1) \quad h_{i_2} \quad a_2 a_3 \ldots a_k \\
 = & \quad \ldots \\
 = & \quad \tau(i_1, a_1)\tau(i_2, a_2)\ldots\tau(i_k, a_k)h_{i_{k+1}}
\end{align*}
\]
\[ h_i s = \tau(i, s) h_{\lambda(i, s)} \]

\[
\begin{align*}
    h_{i_1} a_1 a_2 a_3 \ldots a_k \\
    = \quad \tau(i_1, a_1) \quad h_{i_2} \quad a_2 a_3 \ldots a_k \\
    = \quad \ldots \\
    = \quad \tau(i_1, a_1) \tau(i_2, a_2) \ldots \tau(i_k, a_k) h_{i_{k+1}}
\end{align*}
\]

‘Problem’: \( h_{i_{k+1}} \) need not belong to \( T \).
Groups: Schreier Theorem

\[ G = \langle A \rangle \text{ – a group} \]
\[ H \leq G \]
\[ \{ h_i : i \in I \} \text{ – left coset representatives.} \]
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**Theorem (Schreier 1927)**

\[ H = \langle \{\tau(i, a) : i \in I, a \in A\} \rangle. \]
Groups: Schreier Theorem

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Theorem (Schreier 1927)
\[ H = \langle \{ \tau(i, a) : i \in I, a \in A \} \rangle \].

Corollary

\[ G \] is finitely generated if and only if \( H \) is finitely generated.
Groups: Schreier Theorem

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Theorem (Schreier 1927)

\[ H = \langle \{ \tau(i, a) : i \in I, \ a \in A \} \rangle. \]

Corollary

\[ G \text{ is finitely generated if and only if } H \text{ is finitely generated.} \]

Rewriting mapping:

\[ \phi(i, aw) = \tau(i, a)\phi(\lambda(i, a), w) \]
Groups: Schreier Theorem

$G = \langle A \rangle$ – a group

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$\{ h_i : i \in I \}$ – left coset representatives.

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$H = \langle \{ \tau(i, a) : i \in I, a \in A \} \rangle$.

Corollary

$G$ is finitely generated if and only if $H$ is finitely generated.

Rewriting mapping:

$\phi(i, aw) = \tau(i, a)\phi(\lambda(i, a), w)$

$\phi(w) = \phi(1, w)$. 
\( \mathcal{H} \)-classes

Exactly the same works in monoids for:

▶ maximal subgroups (group \( \mathcal{H} \)-classes);
▶ arbitrary subgroups;
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- Schützenberger groups.
Green’s relations

\[ u \mathcal{L} v \iff Su = Sv \]
\[ u \mathcal{R} v \iff uS = vS \]
\[ u \mathcal{H} v \iff u \mathcal{L} v \& u \mathcal{R} v \]

Definition

The Green index \([S : T]\) of \(T\) is \(S\) is the number of \(H\)-classes in \(S\) \(\setminus T\).
Relative Green’s relations (Wallace 1963)

\[ u \mathcal{L}^T v \iff T u = T v \]
\[ u \mathcal{R}^T v \iff u T = v T \]
\[ u \mathcal{H}^T v \iff u \mathcal{L}^T v \& u \mathcal{R}^T v \]

\((T \leq S)\)
Relative Green’s relations (Wallace 1963)

\[
\begin{align*}
  u \mathcal{L}^T v & \iff Tu = Tv \\
  u \mathcal{R}^T v & \iff uT = vT \\
  u \mathcal{H}^T v & \iff u \mathcal{L}^T v \& u \mathcal{R}^T v \\
  (T \leq S)
\end{align*}
\]

Definition

The Green index \([S : T]_G\) of \(T\) is \(S\) is the number of \(\mathcal{H}^T\)-classes in \(S \setminus T\).
Green, Rees and group indices

\[ (i) | S \setminus T | < \infty \Rightarrow [S : T] G \vartriangleleft \infty. \]

\[ (ii) [G : H] \vartriangleleft \infty \iff [G : H] G \vartriangleleft \infty. \]

Proof (ii)

- \( LH \)-classes = right cosets
- \( RH \)-classes = left cosets
- \( HH \)-classes = intersections
Green, Rees and group indices

Proposition

(i) \(|S \setminus T| < \infty \Rightarrow [S : T]_G < \infty\).
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(i) $|S \setminus T| < \infty \Rightarrow [S : T]_G < \infty$.

(ii) $[G : H] < \infty \Leftrightarrow [G : H]_G < \infty$. 
Green, Rees and group indices

Proposition

(i) \(|S \setminus T| < \infty \Rightarrow [S : T]_G < \infty\).
(ii) \([G : H] < \infty \iff [G : H]_G < \infty\).

Proof

(ii)
\(\mathcal{L}^H\)-classes = right cosets
Proposition

(i) $|S \setminus T| < \infty \Rightarrow [S : T]_G < \infty$.
(ii) $[G : H] < \infty \iff [G : H]_G < \infty$.

Proof

(ii)

$L^H$-classes = right cosets
$R^H$-classes = left cosets
$H^H$-classes = intersections
Green rewriting
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\[ h_i s = \tau(i, s) h^\lambda(i, s) \]
Green rewriting

\[ h_is = \tau(i, s)h_{\lambda(i,s)} \]

\[ \{ h_i : i \in I \} - \text{representatives of } L^T\text{-classes} \]
Green rewriting

\[ h_is = \tau(i, s)h_{\lambda(i,s)} \]

\{ h_i : i \in I \} – representatives of \( \mathcal{L}^T \)-classes, or \( \mathcal{H}^T \)-classes.
Green rewriting

\[ h_is = \tau(i, s)h_{\lambda(i,s)} \]

\[ \{ h_i : i \in I \} \text{ – representatives of } L^T\text{-classes, or } H^T\text{-classes.} \]

\[ h_{i_1}a_1a_2a_3 \ldots a_k \]
\[ = \tau(i_1, a_1)h_{i_2}a_2a_3 \ldots a_k \]
\[ = \ldots \]
\[ = \tau(i_1, a_1)\tau(i_2, a_2) \ldots \tau(i_k, a_k)h_{i_{k+1}} \]
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\[ h_{i_1} a_1 a_2 a_3 \ldots a_k = \tau(i_1, a_1) h_{i_2} a_2 a_3 \ldots a_k = \ldots = \tau(i_1, a_1) \tau(i_2, a_2) \ldots \tau(i_k, a_k) h_{i_{k+1}} \]

\[ \lambda(i, s) : \begin{cases} 
  h_i s \in H_{\lambda(i, s)} & \text{if } h_i s \in S \setminus T \\
  1 & \text{otherwise}
\end{cases} \]
Example

\[ S = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cup \{0\} \]

Multiplication: \((a, b, c) (d, e, f) = \begin{cases} (a, b + e, f) & \text{if } c = d = 0 \\ 0 & \text{otherwise} \end{cases} \)

\[ T = \{ (a, b, c) : a \geq c \} \cup \{0\} \]

\[ S \setminus T = \{ (0, x, 1) : x \in \mathbb{Z} \} \]

\[ (0, y - x, 0)(0, x, 1) = (0, y, 1) \]

Hence \( S \setminus T \) is an \( \mathbb{L} \) class.

\( T \) is not finitely generated, as all \((1, x, 1)\) are indecomposable.

Nik Ruskuc: Rewriting for (sub)semigroups
Example

Set: \( S = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_2 \cup \{0\} \)

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Example

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Example

Set: \( S = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_2 \cup \{0\} \)

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Hence \( S \setminus T \) is an \( \mathcal{L}^T \)-class.

\( T \) is not finitely generated, as all \((1, x, 1)\) are indecomposable.
Green rerewriting 😊

\[ h_i = \tau(i, s) \lambda(i, s) \]
\[ a_1 a_2 a_3 \ldots a_k = \tau(i_1, a_1) \tau(i_2, a_2) \ldots \tau(i_k, a_k) h_{i+1} \]
\[ h_i = \rho(s, i) \sigma(s, i) \tau(i_1, a_1) \tau(i_2, a_2) \ldots \tau(i_k, a_k) h_{j k} = \tau(i_1, a_1) \ldots \tau(i_{k-1}, a_{k-1}) h_{j k-1} \sigma(\tau(i_k, a_k), j k) = \ldots = h_{j 0} \sigma(\tau(i_1, a_1), j 1) \ldots \sigma(\tau(i_{k-1}, a_{k-1}), j k-1) \sigma(\tau(i_k, a_k), j k) \]

Lemma:
If \( a_1 \ldots a_k \in T \) then \( h_{j 0} = 1 \).
Green rerewriting 😊

\[ h_i s = \tau(i, s) h_{\lambda(i, s)} \]

\[ a_1 a_2 a_3 \ldots a_k = \tau(i_1, a_1) \tau(i_2, a_2) \ldots \tau(i_k, a_k) h_{i_{k+1}} \]
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\[ h_is = \tau(i, s)h_{\lambda(i,s)} \]
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\[ \tau(i_1, a_1)\tau(i_2, a_2) \ldots \tau(i_k, a_k) h_{j_k} \]

\[ = \tau(i_1, a_1) \ldots \tau(i_{k-1}, a_{k-1}) h_{\rho(\tau(i_k, a_k), j_k)} \sigma(\tau(i_k, a_k), j_k) \]
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\[ h_is = \tau(i, s)h_{\lambda(i, s)} \]
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\[
\begin{align*}
\tau(i_1, a_1)\tau(i_2, a_2) \ldots \tau(i_k, a_k) & \quad h_{j_k} \\
= & \quad \tau(i_1, a_1) \ldots \tau(i_{k-1}, a_{k-1}) \quad h_{j_{k-1}} \quad \sigma(\tau(i_k, a_k), j_k)
\end{align*}
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= &\quad \ldots \\
= &\quad h_{j_0}\sigma(\tau(i_1, a_1), j_1) \ldots \sigma(\tau(i_{k-1}, a_{k-1}), j_{k-1})\sigma(\tau(i_k, a_k), j_k)
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\end{align*}
\]

**Lemma**

If \( a_1 \ldots a_k \in T \) then \( h_{j_0} = 1 \).
Generation theorems for Green index

Theorem
If \( S = \langle A \rangle \) and \( \{ h_i : i \in I \} \) are representatives of \( H_T \)-classes in \( S \) \( \setminus \) \( T \) then
\[
T = \langle \{ \sigma(\tau(i, a), j) : a \in A, i, j \in I \} \rangle.
\]

Corollary
Suppose \( [S : T]_G < \infty \). Then \( S \) is finitely generated if and only if \( T \) is finitely generated.

Rewriting mapping:
\[
\phi(a_1 a_2 \ldots a_k) = b_{i_1}, a_{i_1}, j_{i_1} b_{i_2}, a_{i_2}, j_{i_2} \ldots b_{i_k}, a_{i_k}, j_{i_k}
\]
+ recursive formulas for \( i_l, j_l \).
Generation theorems for Green index

Theorem

If $S = \langle A \rangle$ and $\{ h_i : i \in I \}$ are representatives of $\mathcal{H}_T$-classes in $S \setminus T$ then

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Theorem

If \( S = \langle A \rangle \) and \( \{ h_i : i \in I \} \) are representatives of \( \mathcal{H}^T \)-classes in \( S \setminus T \) then

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Suppose $[S : T]_G < \infty$. Then $S$ is finitely generated if and only if $T$ is finitely generated.

Rewriting mapping:

$$\phi(a_1a_2 \ldots a_k) = b_{i_1,a_1,j_1} b_{i_2,a_2,j_2} \ldots b_{i_k,a_k,j_k}$$
Theorem

If $S = \langle A \rangle$ and $\{ h_i : i \in I \}$ are representatives of $\mathcal{H}^T$-classes in $S \setminus T$ then

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Rewriting mapping:

$$\phi(a_1 a_2 \ldots a_k) = b_{i_1, a_1} b_{i_2, a_2} b_{i_3, a_3} \ldots b_{i_k, a_k} j_k$$

+ recursive formulas for $i_l, j_l$. 

Do we need $\mathcal{H}^T$-classes?
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Or is sufficient to require:

$$\forall s \in S : \exists i \in I : \exists t, t' \in T : s = th_i = h_it'$$
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Do we need $\mathcal{H}^T$-classes?
Do we need $H^T$-classes?

Example

$M$ – a f.g. monoid
Do we need $\mathcal{H}^T$-classes?

Example

$M$ – a f.g. monoid
$J$ – a non f.g. ideal
Do we need $\mathcal{H}^T$-classes?

**Example**

$M$ – a f.g. monoid

$J$ – a non f.g. ideal

$\overline{M} = \{ \overline{m} : m \in M \}$
Do we need $\mathcal{H}^T$-classes?

**Example**

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$S = M \cup \overline{M}$
Do we need $\mathcal{H}^T$-classes?

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Do we need $\mathcal{H}^T$-classes?

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$S = \langle M \cup \{1_M \} \rangle$
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$
$T = M \cup \overline{J} 
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Do we need $\mathcal{H}^T$-classes?

Example

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$T$ not f.g.  
But: $\overline{m} = 1_M \overline{m} = m1_M$
What are rewriting mappings for?

Typically, they are involved in statements and proofs regarding properties and conditions related to generators. For example...

Theorem

Suppose $S$ is defined by a presentation $\langle A | R \rangle$, and that $T = \langle B \rangle \leq S$. Furthermore, suppose $\phi : L(A, T) \to B^*$ is a rewriting mapping. Then the presentation $\langle B | \phi(\beta) = b, \phi(w_1w_2) = \phi(w_1)\phi(w_2), \phi(w_3uw_4) = \phi(w_3vw_4) \rangle$ defines $T$. 

Nik Ruskuc: Rewriting for (sub)semigroups
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What are rewriting mappings for?

Typically, they are involved in statements and proofs regarding properties and conditions related to generators. For example.

**Theorem**

*Suppose $S$ is defined by a presentation $\langle A \mid R \rangle$, and that $T = \langle B \rangle \leq S$. Furthermore, suppose $\phi : \mathcal{L}(A, T) \rightarrow B^*$ is a rewriting mapping. Then the presentation*

$$\langle B \mid \phi(\beta) = b, \phi(w_1w_2) = \phi(w_1)\phi(w_2), \phi(w_3uw_4) = \phi(w_3vw_4) \rangle$$

*(for $b \in B$, $w_1, w_2, w_3uw_4 \in \mathcal{L}(A, T)$, $(u, v) \in R$)*

*defines $T$.***
Reidemeister–Schreier type theorems

Theorem
Suppose $G < H < \infty$. Then $G$ is finitely presented if and only if $H$ is finitely presented.

Proof
Recall:
\[
\phi(i, aw) = \tau(i, a) \phi(\lambda(1, aw), w),
\]
\[
\phi(w_1 w_2) = \phi(w_1) \phi(w_2).
\]

Theorem
Suppose $|S \setminus T| < \infty$. Then $S$ is finitely presented if and only if $T$ is finitely presented.

Proof
Similar but much harder.
Reidemeister–Schreier type theorems

Theorem

Suppose \([G : H] < \infty\). Then \(G\) is finitely presented if and only if \(H\) is finitely presented.

\[\phi(w_1w_2) = \phi(w_1)\phi(w_2) \quad \phi(w_3uw_4) = \phi(w_3)\phi(\lambda(1,w_3),u)\phi(\lambda(1,w_3u),w_4)\]
Reidemeister–Schreier type theorems

Theorem

Suppose $[G : H] < \infty$. Then $G$ is finitely presented if and only if $H$ is finitely presented.

Proof

Recall: $\phi(i, aw) = \tau(i, a)\phi(\lambda(i, a), w)$, $\phi(w) = \phi(1, w)$.

$\phi(w_1w_2) = \phi(w_1)\phi(w_2)$

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Recall: \(\phi(i, aw) = \tau(i, a)\phi(\lambda(i, a), w), \ \phi(w) = \phi(1, w)\).
\[\phi(w_1w_2) = \phi(w_1)\phi(w_2)\]
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Theorem
Suppose \(|S \setminus T| < \infty\). Then \(S\) is finitely presented if and only if \(T\) is finitely presented.

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Similar but much harder.
Reidemeister–Schreier–Green?

And if it [the solution] was perhaps less sound than I had thought in the first flush of discovery, its inelegance never diminished. And it was above all inelegant in this, to my mind.

(S. Beckett, Molloy)

Problem

If $S$ is finitely presented and $[S:T] < \infty$ is $T$ finitely presented?

What is at issue:
The way $\phi$ is defined (re-re-writing).

Theorem

Suppose $[S:T] < \infty$. If $T$ and all the relative Schützenberger groups are finitely presented then $S$ is finitely presented.
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Suppose $[S : T]_G < \infty$. If $T$ and all the relative Schützenberger groups are finitely presented then $S$ is finitely presented.
But: Reidemeister–Schreier–Malcev–Green 😊

Definition
Suppose \( S \) is embeddable into a group. A presentation \( \langle A \mid R \rangle \) is a Malcev presentation for \( S \) if \( S \simeq A^* / \rho \) where \( \rho \) is the smallest congruence on \( A^* \) such that \( A^* / \rho \) is embeddable into a group.

Theorem
Suppose \( [S : T] \) \( G < \infty \). Then \( S \) has a finite Malcev presentation if and only if \( T \) has a finite Malcev presentation.
Definition

Suppose $S$ is embeddable into a group. A presentation $\langle A \mid R \rangle$ is a **Malcev presentation** for $S$ if $S \cong A^*/\rho$ where $\rho$ is the smallest congruence on $A^*$ such that $A^*/\rho$ is embeddable into a group.
Definition

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Theorem

Suppose $[S : T]_G < \infty$. Then $S$ has a finite Malcev presentation if and only if $T$ has a finite Malcev presentation.
Automatic structures and rewriting

Observation

All the rewriting mappings we have encountered so far can be realised by transducers.

Theorem

If \([S \leq T]\) \(G < \infty\) and \(S\) is automatic then \(T\) is automatic.

Remark

The converse holds for groups and finite Rees index, but not in general.
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\( \Sigma(\rho) \) – the largest congruence on \( S \) contained in \( \rho \)
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If \( G \) is a group and \( H \leq G \) then \( \Sigma(H) \) is the congruence determined by the kernel of \( H \).
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Example

If \( S \) is a semigroup and \( J \) is an ideal, then \( \Sigma(J) \) is the Rees congruence \( (J \times J) \cup \Delta_S \).
Syntactic congruences and Green index

**Theorem**

If $[S : T]_G < \infty$ then $\Sigma(T)$ has finitely many classes.
Syntactic congruences and Green index

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Related to this:

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Suppose \([S : T]_G < \infty\). Then \(S\) is residually finite if and only if \(T\) is residually finite.
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Related to this:

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Suppose $[S : T]_G < \infty$. Then $S$ is residually finite if and only if $T$ is residually finite.

Corollary

Let $S$ be finitely generated, and let $m \in \mathbb{N}$. Then there exist only finitely many subsemigroups $T \leq S$ such that $[S : T]_G \leq m$. 
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Corollary
Let \(S\) be finitely generated, and let \(m \in \mathbb{N}\). Then there exist only finitely many subsemigroups \(T \leq S\) such that \([S : T]_G \leq m\).

Proof
There are only finitely many actions of \(S\) on a finite set.
A surprising undecidability result (after Jura 1980)

Theorem
There is no algorithm which has as its input a finite presentation \( \langle A | R \rangle \) defining a semigroup \( S \) and a number \( n \in \mathbb{N} \) and which outputs generating sets of all subsemigroups of \( S \) of Green index \( \leq n \).

Remark
Such an algorithm exists for groups, and has practical implementations.

Proof
Suppose otherwise. \( \{0\} \) has Green index \( |S| \) in \( S \). Run algorithm for \( n = 1 \), and see if \( \{0\} \) is among the outputs. This decides triviality of \( S \), a contradiction.
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There is no algorithm which has as its input a finite presentation \langle A \mid R \rangle defining a semigroup $S$ and a number $n \in \mathbb{N}$ and which outputs generating sets of all subsemigroups of $S$ of Green index $\leq n$.

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Such an algorithm exists for groups, and has practical implementations.

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