Rewriting Generators for (Sub)Semigroups

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University
of
St Andrews

Setting the Scene

S – a semigroup/monoid

$S = \langle A \rangle$

$T \leq S$

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S – a semigroup/monoid

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Find a generating set for T.

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Find a 'nice' generating set for T.

People

- Robert Gray, NR, Green index and finiteness conditions for semigroups, J. Algebra 320 (2008), 3145–3164.
- Alan J. Cain, Robert Gray, NR, Green index in semigroups: generators and related finiteness conditions, about to be submitted.
- Also: C.M. Campbell, M. Hoffmann, E.F. Robertson, R.M Thomas.
- Related: R. Gray, V. Maltcev, J.D. Mitchell, NR, Ideals, finiteness conditions and Green index for subsemigroups, about to be submitted.

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Given a 'long' word from A^* representing an element of T, 'decompose' it into 'shorter' members of T.

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Hence $T = \langle a^2, ab, ba, b^2 \rangle$.

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Theorem (Jura 1978)

If $S = \langle A \rangle$ *then*

$$T = \langle \{ uav \ : \ u, v \in (S \setminus T)^1, \ a \in A, \ ua, uav \in T \} \rangle.$$

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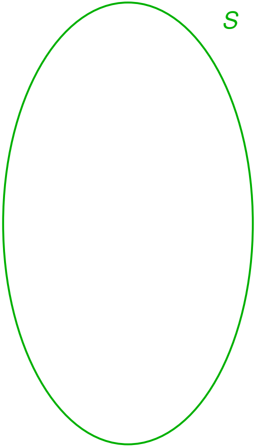
From the previous slides: For $w \in \mathcal{L}(A, T)$ let $w = uav$, where ua is the shortest prefix in $\mathcal{L}(A, T)$. Then

$$\phi(w) = \begin{cases} (ua) \cdot \phi(v) & \text{if } v \in \mathcal{L}(A, T) \\ w & \text{otherwise.} \end{cases}$$

Another idea: ‘Coset representatives’



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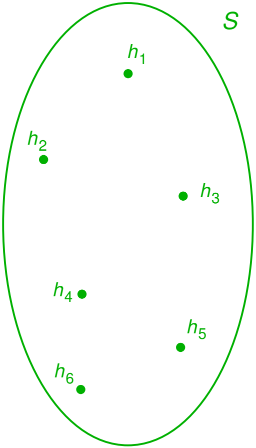

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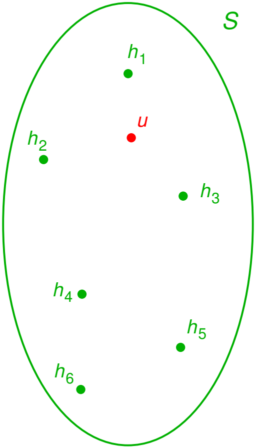

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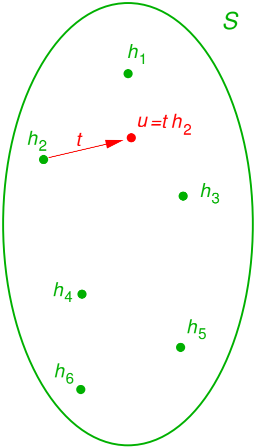

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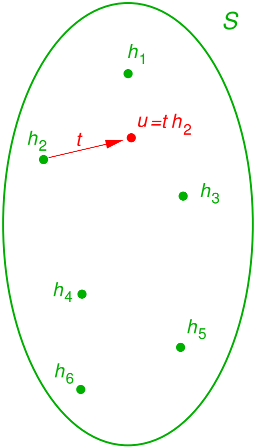

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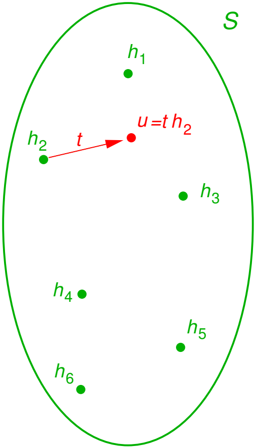

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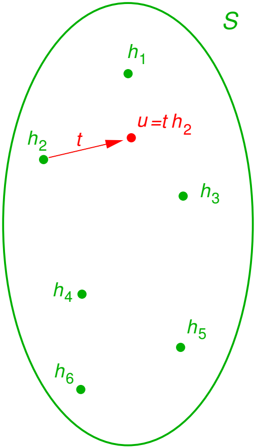

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$\tau(i, s) \in T,\ \lambda(i, s) \in I$

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$$h_{i_1} a_1 a_2 a_3 \dots a_k$$

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$$
\begin{aligned}
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= \;& \tau(i_1, a_1) h_{\lambda(i_1,a_1)} a_2 a_3 \dots a_k
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'Problem': $h_{i_{k+1}}$ need not belong to T.

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Green's relations

$$
\begin{aligned}
u\,\mathcal{L}\;\; v &\Leftrightarrow Su = Sv\\
u\,\mathcal{R}\;\; v &\Leftrightarrow uS = vS\\
u\,\mathcal{H}\;\; v &\Leftrightarrow u\,\mathcal{L}\;\; v \;\&\; u\,\mathcal{R}\;\; v
\end{aligned}
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Relative Green's relations (Wallace 1963)

$$u\,\mathcal{L}^T v \;\Leftrightarrow\; Tu = Tv$$
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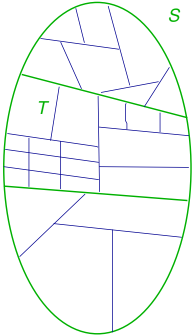

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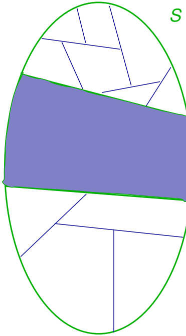

Definition

The Green index $[S:T]_G$ of T is S is the number of $\mathcal{H}^T$-classes in $S \setminus T$.

Green, Rees and group indices

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Proposition

(i) $|S \setminus T| < \infty \Rightarrow [S : T]_G < \infty$.

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$\mathcal{L}^H$-classes = right cosets

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$\mathcal{H}^H$-classes = intersections

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$$\lambda(i,s) : \begin{cases} h_i s \in H_{\lambda(i,s)} & \text{if } h_i s \in S \setminus T \\ 1 & \text{otherwise} \end{cases}$$

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Set: $S = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_2 \cup \{0\}$

Multiplication: $(a, b, c)(d, e, f) = \begin{cases} (a, b + e, f) & \text{if } c = d = 0 \\ 0 & \text{otherwise} \end{cases}$

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T is not finitely generated, as all $(1, x, 1)$ are indecomposable.

Green rerewriting

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Lemma

If $a_1 \dots a_k \in T$ then $h_{j_0} = 1$.

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If $S = \langle A \rangle$ and $\{h_i : i \in I\}$ are representatives of $\mathcal{H}^T$-classes in $S \setminus T$ then

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$\phi(a_1 a_2 \dots a_k) = b_{i_1,a_1,j_1} b_{i_2,a_2,j_2} \dots b_{i_k,a_k,j_k}$

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$+$ recursive formulas for i_l, j_l.

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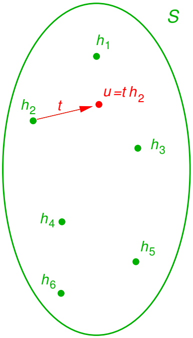

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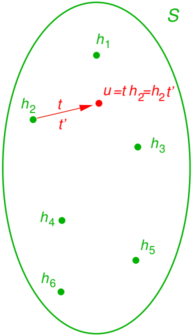

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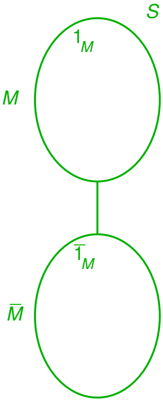

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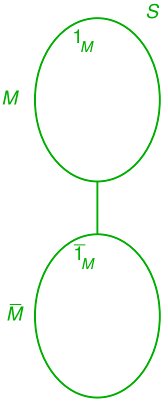

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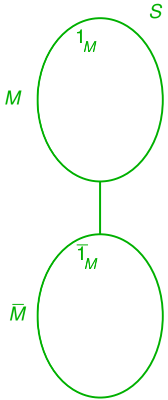

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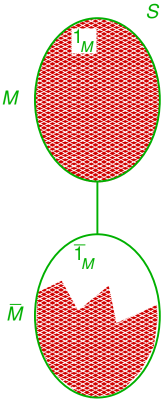

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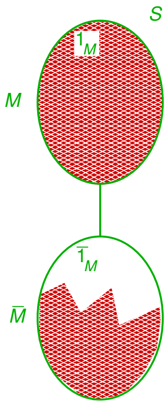

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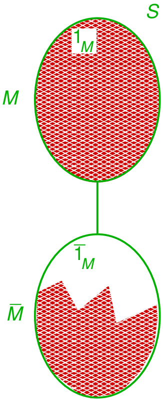

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For example...

Theorem

Suppose S is defined by a presentation $\langle A \mid R\rangle$, and that $T = \langle B\rangle \leq S$. Furthermore, suppose $\phi : \mathcal{L}(A, T) \to B^*$ is a rewriting mapping. Then the presentation

$$\begin{aligned} \langle B \quad \mid \quad & \phi(\beta) = b, \\ & \phi(w_1 w_2) = \phi(w_1)\phi(w_2), \\ & \phi(w_3 u w_4) = \phi(w_3 v w_4) \\ & (b \in B,\ w_1, w_2, w_3 u w_4 \in \mathcal{L}(A, T),\ (u, v) \in R)\rangle \end{aligned}$$

defines T.

Reidemeister–Schreier type theorems

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Suppose $[G : H] < \infty$. Then G is finitely presented if and only if H is finitely presented.

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Proof

Recall: $\phi(i, aw) = \tau(i, a)\phi(\lambda(i, a), w)$, $\phi(w) = \phi(1, w)$.

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Suppose $|S \setminus T| < \infty$. Then S is finitely presented if and only if T is finitely presented.

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Similar but much harder.

Reidemeister–Schreier–Green?

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And if it [the solution] was perhaps less sound than I had thought in the first flush of discovery, its inelegance never diminished. And it was above all inelegant in this, to my mind... (S. Beckett, Molloy)

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If S is finitely presented and $[S : T]_G < \infty$ is T finitely presented?

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Suppose $[S:T]_G < \infty$. If T and all the relative Schützenberger groups are finitely presented then S is finitely presented.

But: Reidemeister–Schreier–Malcev–Green

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Definition

Suppose S is embeddable into a group. A presentation $\langle A \mid R\rangle$ is a Malcev presentation for S if $S \cong A^*/\rho$ where ρ is the smallest congruence on A^* such that A^*/ρ is embeddable into a group.

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Theorem

Suppose $[S : T]_G < \infty$. Then S has a finite Malcev presentation if and only if T has a finite Malcev presentation.

Automatic structures and rewriting

Obervation

All the rewriting mappings we have encountered so far can be realised by transducers.

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If $[S : T]_G < \infty$ *and* S *is automatic then* T *is automatic.*

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Remark

The converse holds for groups and finite Rees index, but not in general.

Syntactic congruences

ρ – an equivalence on a semigroup S

$\Sigma(\rho)$ – the largest congruence on S contained in ρ

$\Sigma(T)$ (for $T \leq S$) $= \Sigma((T \times T) \cup ((S \setminus T) \times (S \setminus T)))$

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Example

If S is a semigroup and J is an ideal, then $\Sigma(J)$ is the Rees congruence $(J \times J) \cup \Delta_S$.

Syntactic congruences and Green index

Theorem

If $[S : T]_G < \infty$ *then* $\Sigma(T)$ *has finitely many classes.*

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Related to this:

Theorem

Suppose $[S : T]_G < \infty$. *Then* S *is residually finite if and only if* T *is residually finite.*

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Suppose $[S : T]_G < \infty$*. Then* S *is residually finite if and only if* T *is residually finite.*

Corollary

Let S *be finitely generated, and let* $m \in \mathbb{N}$*. Then there exist only finitely many subsemigroups* $T \leq S$ *such that* $[S : T]_G \leq m$*.*

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Proof

There are only finitely many actions of S on a finite set.

A surprising undecidability result (after Jura 1980)

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This decides triviality of S, a contradiction.