Rewriting Generators for (Sub)Semigroups

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Porto, 6 July 2009

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Setting the Scene

S – a semigroup/monoid

$S = \langle A \rangle$

$T \leq S$

Find a generating set for T.

People

- Robert Gray, NR, Green index and finiteness conditions for semigroups, J. Algebra 320 (2008), 3145–3164.
- Alan J. Cain, Robert Gray, NR, Green index in semigroups: generators and related finiteness conditions, about to be submitted.
- Also: C.M. Campbell, M. Hoffmann, E.F. Robertson, R.M Thomas.
- Related: R. Gray, V. Maltcev, J.D. Mitchell, NR, Ideals, finiteness conditions and Green index for subsemigroups, about to be submitted.

A first naive approach

Given a 'long' word from A^* representing an element of T, 'decompose' it into 'shorter' members of T.

$$a_1 a_2 a_3 | a_4 a_5 | a_6 | a_7 a_8 a_9 | \dots$$

$$\notin T \qquad\qquad \in T$$

Example

$S = \{a, b\}^*$

$T = \{\text{words of even length}\}$

$$ab|aa|ba|bb$$

Hence $T = \langle a^2, ab, ba, b^2 \rangle$.

Variation: finite complement (Rees index)

Suppose $|S \setminus T| = r < \infty$

Consider a product $a_1 a_2 \dots a_{r+1}$. Then:

(i) $\exists k \ : \ a_1 \dots a_k \in T$, or

(ii) $\exists k, l \ : \ a_1 \dots a_k = a_1 \dots a_l$.

Theorem (Jura 1978)

If $S = \langle A \rangle$ then

$$T = \langle \{ uav \ : \ u, v \in (S \setminus T)^1, \ a \in A, \ ua, uav \in T \} \rangle.$$

Corollary

S is finitely generated if and only if T is finitely generated.

Rewriting mapping

Suppose $S = \langle A \rangle,\ T \leq S,\ T = \langle B \rangle$.

$\mathcal{L}(A, T) = \{$words from A^* representing elements of $T\}$

Then there exists a mapping

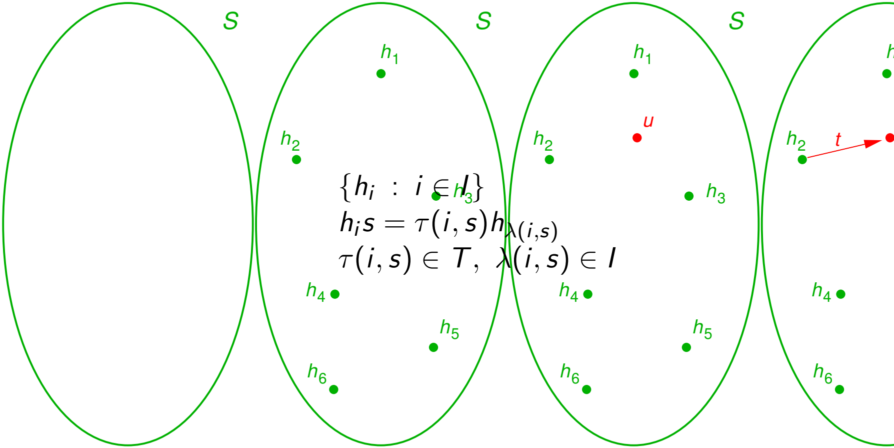
$$\phi \,:\, \mathcal{L}(A, T) \rightarrow B^*$$

such that $w \in \mathcal{L}(A, T)$ and $\phi(w) \in B^*$ represent the same element of T.

From the previous slides: For $w \in \mathcal{L}(A, T)$ let $w = uav$, where ua is the shortest prefix in $\mathcal{L}(A, T)$. Then

$$\phi(w) = \begin{cases} (ua) \cdot \phi(v) & \text{if } v \in \mathcal{L}(A, T) \\ w & \text{otherwise.} \end{cases}$$

Another idea: 'Coset representatives'



$\{h_i \,:\, i \in I\}$

$h_i s = \tau(i,s) h_{\lambda(i,s)}$

$\tau(i,s) \in T,\ \lambda(i,s) \in I$

$h_i s = \tau(i, s) h_{\lambda(i,s)}$

$$
\begin{aligned}
& 1\ a_1 a_2 a_3 \dots a_k \\
= \ & \tau(i_1, a_1) h_{\lambda(i_1,a_1)} a_2 a_3 \dots a_k \\
= \ & \dots \\
= \ & \tau(i_1, a_1) \tau(i_2, a_2) \dots \tau(i_k, a_k) h_{i_{k+1}}
\end{aligned}
$$

'Problem': $h_{i_{k+1}}$ need not belong to T.

Groups: Schreier Theorem

$G = \langle A \rangle$ – a group

$H \leq G$

$\{h_i \ : \ i \in I\}$ – left coset representatives.

Theorem (Schreier 1927)

$H = \langle \{\tau(i, a) \ : \ i \in I, \ a \in A\} \rangle$.

Corollary

G is finitely generated if and only if H is finitely generated.

Rewriting mapping:

$$\begin{aligned} \phi(i, aw) &= \tau(i, a)\phi(\lambda(i, a), w) \\ \phi(w) &= \phi(1, w). \end{aligned}$$

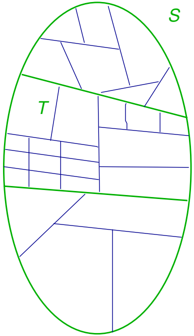
$\mathcal{H}$-classes

Exactly the same works in monoids for:

- maximal subgroups (group $\mathcal{H}$-classes);
- arbitrary subgroups;
- Schützenberger groups.

Relative Green's relations (Wallace 1963)

$$u\,\mathcal{L}^T v \;\Leftrightarrow\; Su = Sv$$
$$u\,\mathcal{R}^T v \;\Leftrightarrow\; uS = vS$$
$$u\,\mathcal{H}^T v \;\Leftrightarrow\; u\,\mathcal{L}^T v \;\&\; u\,\mathcal{R}^T v$$
$$(T \leq S)$$



Definition

The Green index $[S : T]_G$ of T is S is the number of $\mathcal{H}^T$-classes in $S \setminus T$.

Green, Rees and group indices

Proposition

(i) $|S \setminus T| < \infty \Rightarrow [S : T]_G < \infty$.

(ii) $[G : H] < \infty \Leftrightarrow [G : H]_G < \infty$.

Proof

(ii)

$\mathcal{L}^H$-classes = right cosets

$\mathcal{R}^H$-classes = left cosets

$\mathcal{H}^H$-classes = intersections

Green rewriting

$h_i s = \tau(i,s) h_{\lambda(i,s)}$

$\{h_i \,:\, i \in I\}$ – representatives of $\mathcal{L}^T$-classes, or $\mathcal{H}^T$-classes.

$$\begin{aligned} & h_{i_1} a_1 a_2 a_3 \dots a_k \\ = \;& \tau(i_1, a_1) h_{i_2} a_2 a_3 \dots a_k \\ = \;& \dots \\ = \;& \tau(i_1, a_1)\tau(i_2, a_2) \dots \tau(i_k, a_k) h_{i_{k+1}} \end{aligned}$$

$$\lambda(i,s) : \begin{cases} h_i s \in H_{\lambda(i,s)} & \text{if } h_i s \in S \setminus T \\ 1 & \text{otherwise} \end{cases}$$

Example

Set: $S = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_2 \cup \{0\}$

Multiplication: $(a, b, c)(d, e, f) = \begin{cases} (a, b+e, f) & \text{if } c = d = 0 \\ 0 & \text{otherwise} \end{cases}$

$T = \{(a, b, c) \ : \ a \geq c\} \cup \{0\}$

$S \setminus T = \{(0, x, 1) \ : \ x \in \mathbb{Z}\}$

$(0, y - x, 0)(0, x, 1) = (0, y, 1)$

Hence $S \setminus T$ is an $\mathcal{L}^T$-class.

T is not finitely generated, as all $(1, x, 1)$ are indecomposable.

Green rerewriting

$$h_i s = \tau(i,s) h_{\lambda(i,s)}$$
$$a_1 a_2 a_3 \dots a_k = \tau(i_1,a_1)\tau(i_2,a_2)\dots\tau(i_k,a_k) h_{i_{k+1}}$$
$$s h_i = h_{\rho(s,i)} \sigma(s,i)$$

$$\begin{aligned} & \tau(i_1,a_1)\tau(i_2,a_2)\dots\tau(i_k,a_k)h_{i_{k+1}} \\ = \; & \tau(i_1,a_1)\dots\tau(i_{k-1},a_{k-1})\, h_{\rho(\tau(i_k,a_k),j_k)}\, \sigma(\tau(i_k,a_k),j_k) \\ = \; & \dots \\ = \; & h_{j_0}\sigma(\tau(i_1,a_1),j_1)\dots\sigma(\tau(i_{k-1},a_{k-1}),j_{k-1})\sigma(\tau(i_k,a_k),j_k) \end{aligned}$$

Lemma

If $a_1 \dots a_k \in T$ then $h_{j_0} = 1$.

Generation theorems for Green index

Theorem

If $S = \langle A \rangle$ *and* $\{h_i \,:\, i \in I\}$ *are representatives of* $\mathcal{H}^T$*-classes in* $S \setminus T$ *then*

$$T = \langle \{\sigma(\tau(i, a), j) \,:\, a \in A,\ i, j \in I\} \rangle.$$

Corollary

Suppose $[S : T]_G < \infty$*. Then* S *is finitely generated if and only if* T *is finitely generated.*

Rewriting mapping:

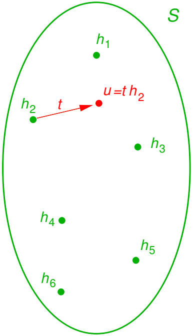
$\phi(a_1 a_2 \dots a_k) = b_{i_1,a_1,j_1} b_{i_2,a_2,j_2} \dots b_{i_k,a_k,j_k}$

$+$ recursive formulas for i_l, j_l.

Do we need $\mathcal{H}^T$-classes?

Or is sufficient to require:

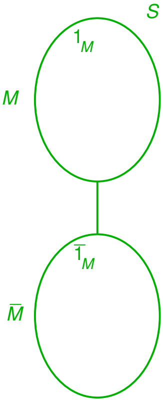
$$\forall s \in S \ : \ \exists i \in I \ : \ \exists t, t' \in T \ : \ s = th_i = h_i t'?$$



Do we need $\mathcal{H}^T$-classes?

Example

M – a f.g. monoid
J – a non f.g. ideal
$\overline{M} = \{\overline{m} \, : \, m \in M\}$
$S = M \dot\cup \overline{M}$
$m\overline{n} = \overline{mn} = \overline{m}n$
$S = \langle M \cup \{\overline{1_M}\} \rangle$
$T = M \cup \overline{J}$
T not f.g.
But: $\overline{m} = \overline{1_M} m = m \overline{1_M}$



What are rewriting mappings for?

Typically, they are involved in statements and proofs regarding properties and conditions related to generators.

For example...

Theorem

Suppose S is defined by a presentation $\langle A \mid R \rangle$, and that $T = \langle B \rangle \leq S$. Furthermore, suppose $\phi : \mathcal{L}(A, T) \to B^*$ is a rewriting mapping. Then the presentation

$$\begin{aligned} \langle B \quad | \quad & \phi(\beta) = b, \\ & \phi(w_1 w_2) = \phi(w_1)\phi(w_2), \\ & \phi(w_3 u w_4) = \phi(w_3 v w_4) \\ & (b \in B,\ w_1, w_2, w_3 u w_4 \in \mathcal{L}(A, T),\ (u, v) \in R) \rangle \end{aligned}$$

defines T.

Reidemeister–Schreier type theorems

Theorem

Suppose $[G : H] < \infty$. Then G is finitely presented if and only if H is finitely presented.

Proof

Recall: $\phi(i, aw) = \tau(i, a)\phi(\lambda(i, a), w)$, $\phi(w) = \phi(1, w)$.

$\phi(w_1 w_2) = \phi(w_1)\phi(w_2)$

$\phi(w_3 u w_4) = \phi(w_3)\phi(\lambda(1, w_3), u)\phi(\lambda(1, w_3 u), w_4)$

Theorem

Suppose $|S \setminus T| < \infty$. Then S is finitely presented if and only if T is finitely presented.

Proof

Similar but much harder.

Reidemeister–Schreier–Green?

And if it [the solution] was perhaps less sound than I had thought in the first flush of discovery, its inelegance never diminished. And it was above all inelegant in this, to my mind. . . (S. Beckett, Molloy)

Problem

If S is finitely presented and $[S : T]_G < \infty$ is T finitely presented?

What is at issue: The way ϕ is defined (re-re-writing).

Theorem

Suppose $[S : T]_G < \infty$. If T and all the relative Schützenberger groups are finitely presented then S is finitely presented.

But: Reidemeister–Schreier–Malcev–Green

Definition

Suppose S is embeddable into a group. A presentation $\langle A \mid R \rangle$ is a Malcev presentation for S if $S \cong A^*/\rho$ where ρ is the smallest congruence on A^* such that A^*/ρ is embeddable into a group.

Theorem

Suppose $[S : T]_G < \infty$. Then S has a finite Malcev presentation if and only if T has a finite Malcev presentation.

Automatic structures and rewriting

Obervation

All the rewriting mappings we have encountered so far can be realised by transducers.

Theorem

If $[S : T]_G < \infty$ *and* S *is automatic then* T *is automatic.*

Remark

The converse holds for groups and finite Rees index, but not in general.

Syntactic congruences

ρ – an equivalence on a semigroup S

$\Sigma(\rho)$ – the largest congruence on S contained in ρ

$\Sigma(T)$ (for $T \leq S$) $= \Sigma((T \times T) \cup ((S \setminus T) \times (S \setminus T)))$

Example

If G is a group and $H \leq G$ then $\Sigma(H)$ is the congruence determined by the kernel of H.

Example

If S is a semigroup and J is an ideal, then $\Sigma(J)$ is the Rees congruence $(J \times J) \cup \Delta_S$.

Syntactic congruences and Green index

Theorem

If $[S : T]_G < \infty$ *then* $\Sigma(T)$ *has finitely many classes.*

Related to this:

Theorem

Suppose $[S : T]_G < \infty$. *Then* S *is residually finite if and only if* T *is residually finite.*

Corollary

Let S *be finitely generated, and let* $m \in \mathbb{N}$. *Then there exist only finitely many subsemigroups* $T \leq S$ *such that* $[S : T]_G \leq m$.

Proof

There are only finitely many actions of S on a finite set.

A surprising undecidability result (after Jura 1980)

Theorem

There is no algorithm which has as its input a finite presentation $\langle A \mid R \rangle$ defining a semigroup S and a number $n \in \mathbb{N}$ and which outputs generating sets of all subsemigroups of S of Green index $\leq n$.

Remark

Such an algorithm exists for groups, and has practical implementations.

Proof

Suppose otherwise.

$\{0\}$ has Green index $|S|$ in S^0.

Run algorithm for $n = 1$, and see if $\{0\}$ is among the outputs.

This decides triviality of S, a contradiction.