Grid Pattern Classes

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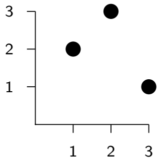

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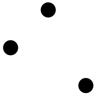

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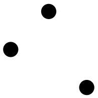

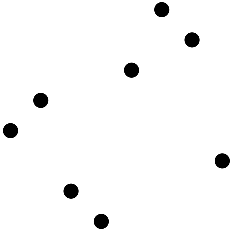

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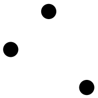

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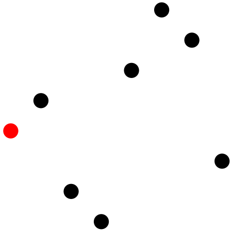

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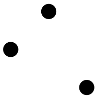

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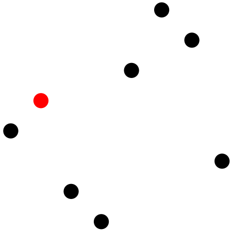

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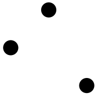

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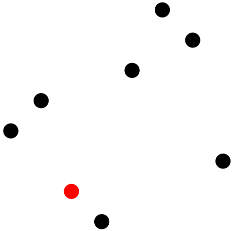

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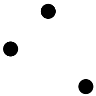
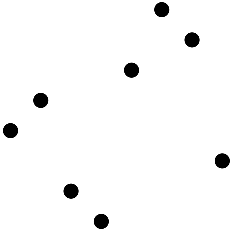


$4, 5, 2, 1, 6, 8, 7, 3$



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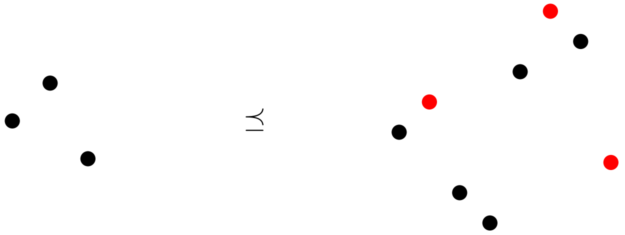

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- If $\sigma \npreceq \tau$ say τ avoids σ.

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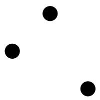
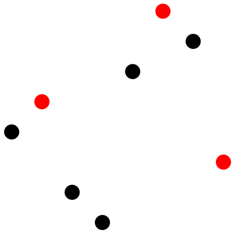
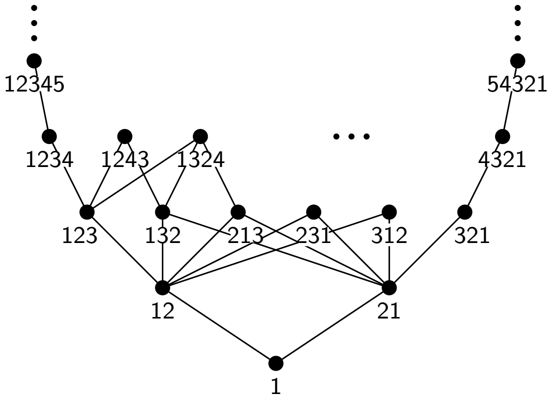


$\preceq$



Poset of permutations under involvement



Pattern classes & bases

- Pattern class = a collection $\mathcal{P}$ of permutations closed downwards under $\preceq$:

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- Not every pattern class is finitely based.

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- Order theoretic properties: antichains, join property,...
- Structure: of permutations in $\mathcal{P}$; of $\mathcal{P}$ itself.

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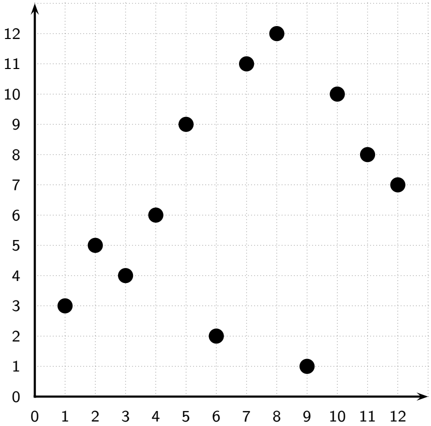

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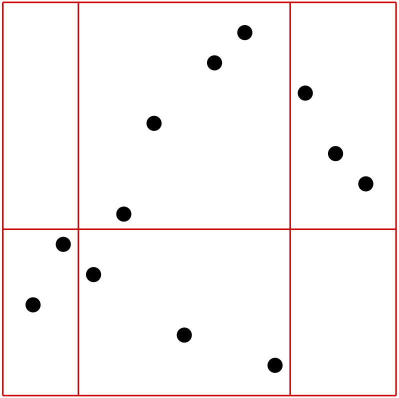

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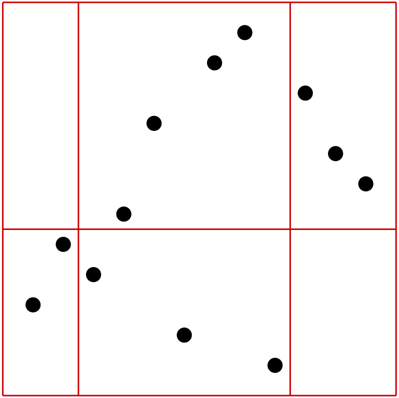

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

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- Permutation π admits an M-gridding if the xy-plane with the graph Γ of π plotted in it can be partitioned into an axis parallel rectangular grid C_{ij} ($i \in [p]$, $j \in [q]$) such that

$$\Gamma \cap C_{ij} \text{ is } \begin{cases} \text{increasing,} & \text{if } m_{ij} = 1 \\ \text{decreasing,} & \text{if } m_{ij} = -1 \\ \emptyset, & \text{if } m_{ij} = 0. \end{cases}$$

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- The (pattern class) of all π that admit an M-gridding is called a (monotone) grid class, and denoted $\mathrm{Grid}(M)$.

Outline & credits

- Grid classes have, implicitly or explicitly, been used as a structural tool in the theory of pattern classes.
- New project: Develop a systematic theory of grid classes.
- Collaborators: Michael Albert (Otago), Mike Atkinson (Otago), Mathilde Bouvel (Bordeaux), Robert Brignall (OU), Vincent Vatter (Florida), NR.

- Methodology: permutations, geometry, automata/languages.

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Matrix: $\begin{pmatrix} -1 & 1 \end{pmatrix}$

Description: juxtapositions of a decreasing and an increasing sequence.

Basis: 231, 132.

Enumeration: 2^{n-1}.

Further examples (1)

Example

Atkinson (1999) proves

$$\mathrm{Av}(321, 2143) = \mathrm{Grid}\begin{pmatrix} 1 & 1 \end{pmatrix} \cup \mathrm{Grid}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and derives as a consequence the enumeration for $\mathrm{Av}(321, 2143)$:

$$2^{n+1} - 2n - 1 - \binom{n+1}{3}.$$

Further examples (2)

Example

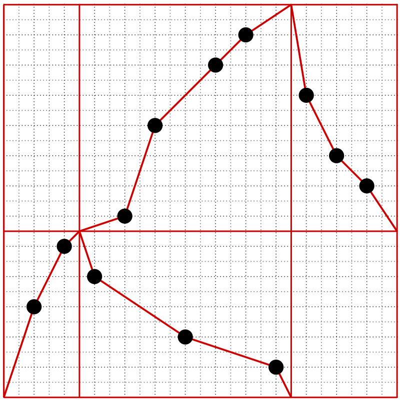
Murphy (2003) expresses several classes with two basis permutations of lengths 3 and 4 as grid classes. For instance

$$\mathrm{Av}(132, 4312) = \mathrm{Grid}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Other occurrences in literature

- Atkinson (1998) – skew merge permutations.
- Profile classes (Atkinson (1999)) – permutation matrices.
- W-classes (Atkinson, Murphy, NR (2002)) – $1 \times q$ matrices.
- Murphy, Vatter (2002) – PWO.
- Waton (2007) – atomicity, Grid $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
- Huczynska, Vatter (2006) – sparse matrices; decidability for subclasses of grid classes.
- Vatter (submitted) – small growth rates.

Playing with the picture



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Definition

If $\Gamma(M)$ is a forest we say that $\mathrm{Grid}(M)$ is a forest grid class.

Graph $\Gamma(M)$: examples

Example

	1	-1
1	-1	

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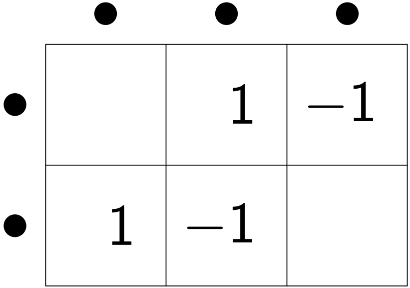

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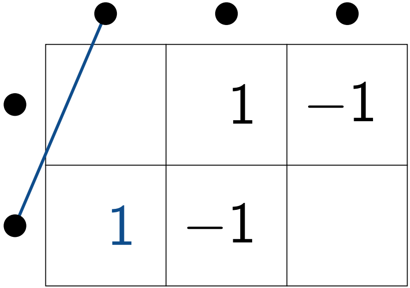

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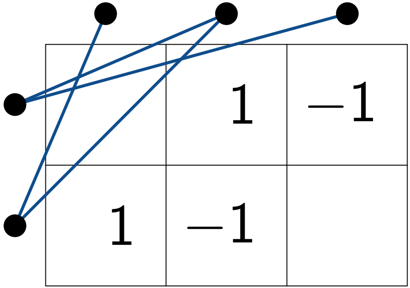

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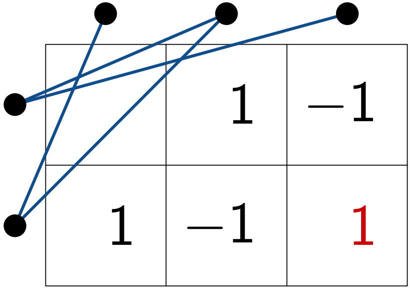

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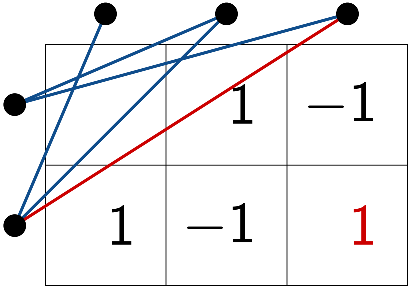

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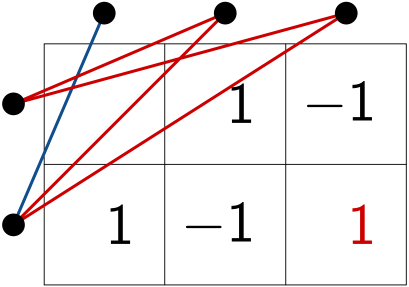

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Theorem (Murphy, Vatter (2002))

A grid class is PWO iff it is a forest grid class.

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- Waton (et al.): permutations from parallel lines; circle permutations; convex permutations; etc.

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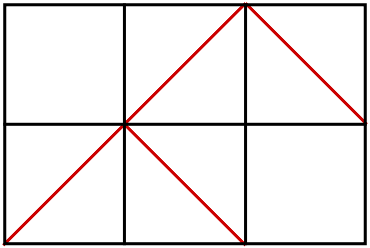

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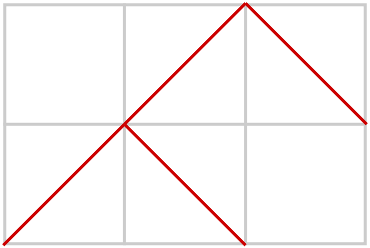

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Finite basis (in words)

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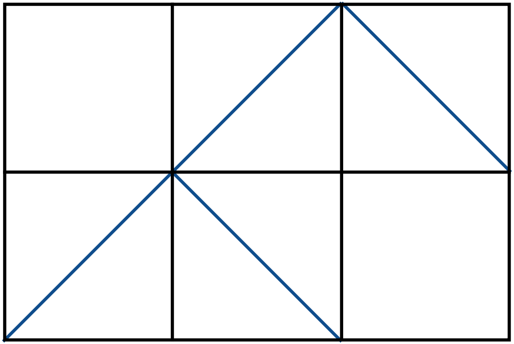

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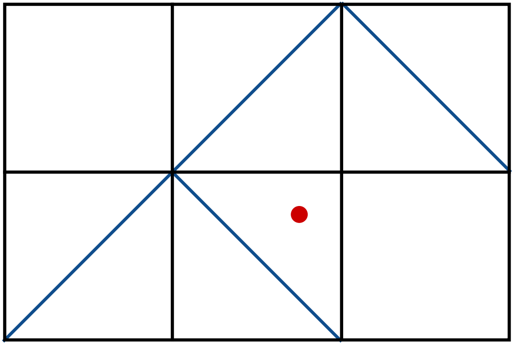

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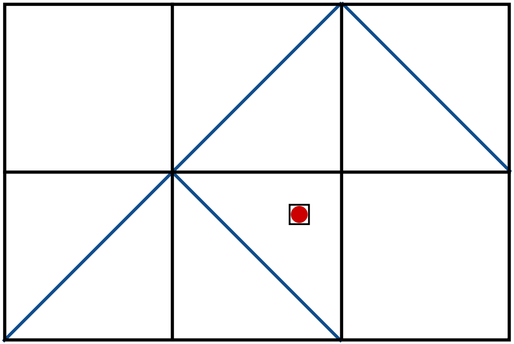

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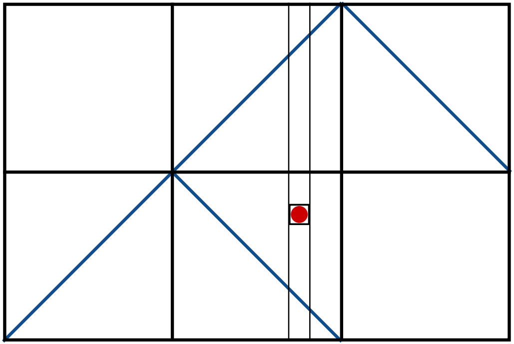

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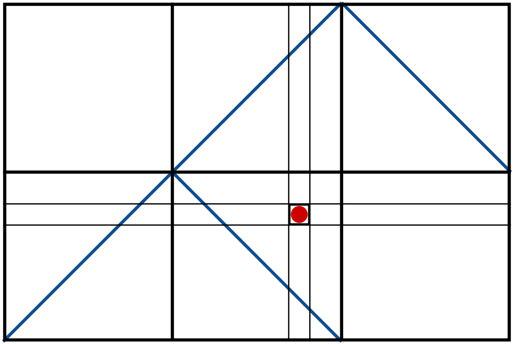

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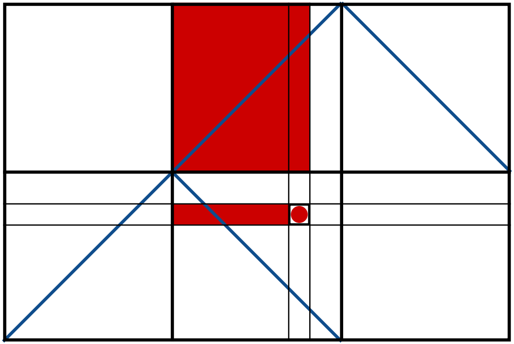

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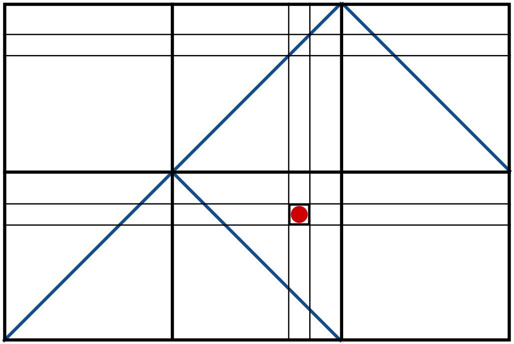

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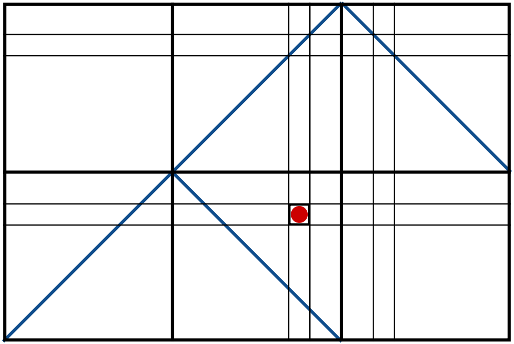

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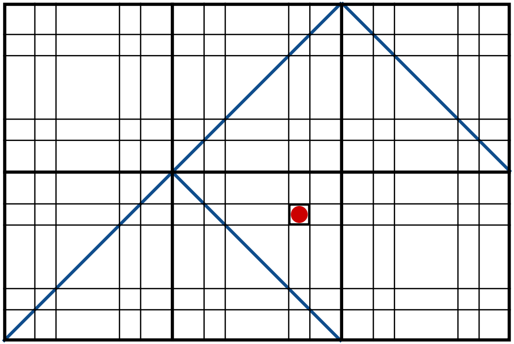

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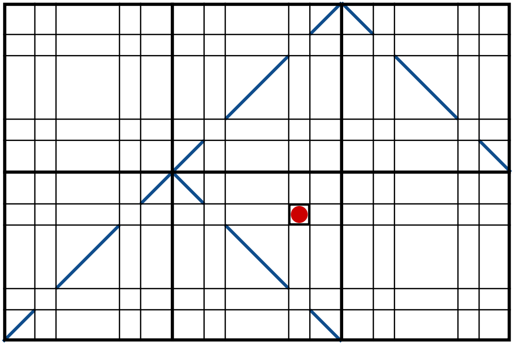

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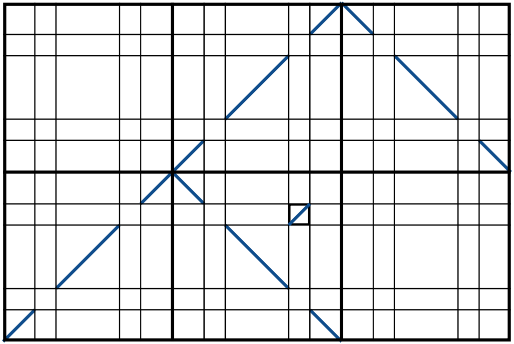

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0	0	0	0	0	0	0	1	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	-1
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	−1
0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
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Corollary

Every subclass of a forest grid class is finitely based.

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- Every closed subset of A^* has a rational generating function.

Encoding gridded permutations as words

Decoding words

- The 'inverse' process.
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Remark

Here ‘forest grid class’ stands for a small generalisation: one point cells are also allowed.

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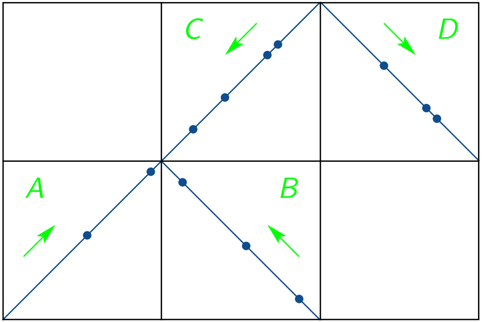

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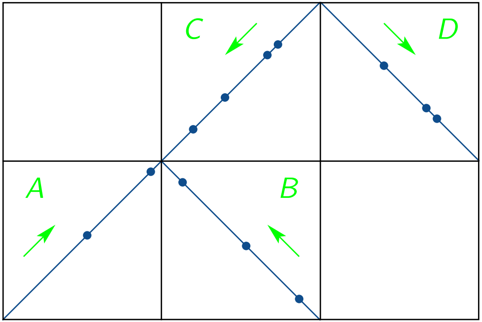

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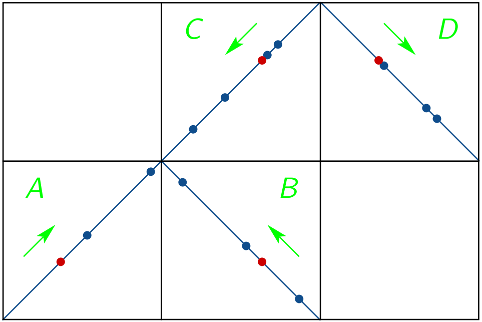

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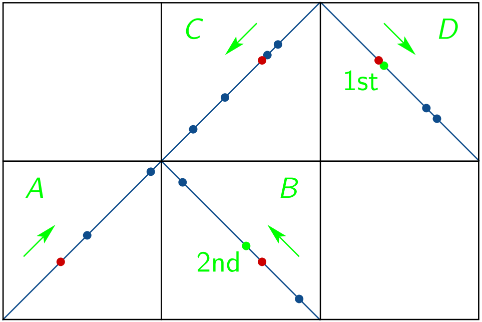

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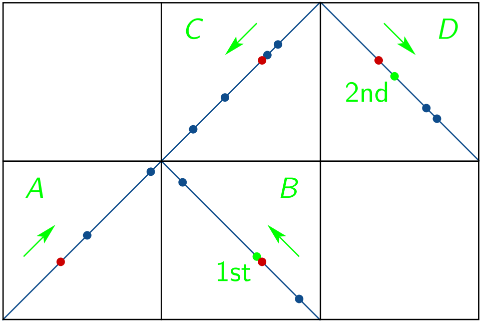

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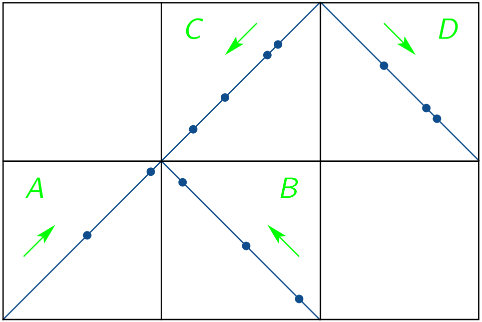

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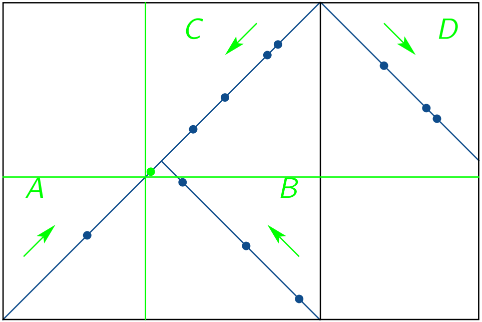

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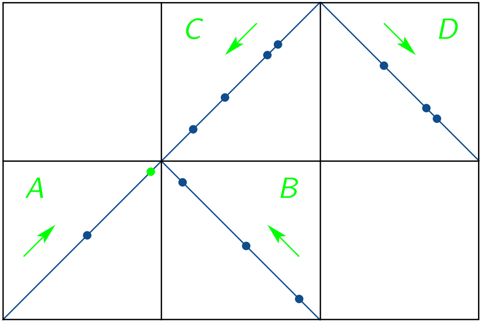

Resolving ambiguities

Independent cells:

- $a_{ij}a_{st} = a_{st}a_{ij}$ where $i \neq s$, $j \neq t$.
- Trace monoids, regular sets of representatives with uniqueness.

Multiple griddings:

- There are only finitely many different gridline movements, relative to the geometric representation.
- An argument similar to basis and subclasses.

Corollary

Every forest grid class has a rational generating function.

Summary of results

For a forest grid class Grid(M) the following hold:

- Grid($\mathcal{M}$) is finitely based.
- Grid($\mathcal{M}$) is partially well ordered.
- Every subclass of Grid($\mathcal{M}$) is a finite union of (slightly generalised) forest grid classes.
- Grid($\mathcal{M}$) and each of its subclasses have rational generating functions.
- The basis and the generating function for Grid($\mathcal{M}$) can be effectively computed from $\mathcal{M}$.
- The sets of $\oplus$-indecomposable, $\ominus$-indecomposable and simple permutations in Grid(M) all have rational generating functions.

Out of the woods?

Conjecture

Every grid class is finitely based.

Conjecture

Every grid class has an algebraic generating function.

Question

Are the basis and the generating function for a grid class algorithmically computable from the gridding matrix?

Question

Is there an algorithm which decides whether a finitely based pattern class (given by its basis) is a grid class?

Evidence

- Waton (2007): Grid $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is finitely based.
- Stankova (1994), Atkinson (1998): Grid $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ has basis $\{2143, 3412\}$ and generating function

$$\frac{1-3x}{(1-2x)\sqrt{1-4x}}.$$

Detecting 'griddable' classes

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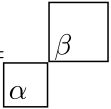
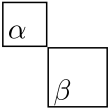
Two constructions: $\alpha \oplus \beta =$ 

, $\alpha \ominus \beta =$ 

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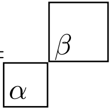
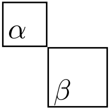
Two constructions: $\alpha \oplus \beta =$ 

, $\alpha \ominus \beta =$ 

Theorem (Huczynska, Vatter 2006)

A pattern class $\mathcal{C}$ is a subclass of a grid class iff its basis contains subpermutations of

$$\underbrace{21 \oplus \cdots \oplus 21}_{m},\ \underbrace{12 \ominus \cdots \ominus 12}_{n}$$

for some m, n.

Further directions

Generalised grid classes: Allow cells in the matrix to contain arbitrary pattern classes.

V. Vatter, Small permutation classes, submitted.

R. Brignall, Grid classes and partial well order, submitted.

Thank you!