Some Combinatorial Properties of Direct Products of Groups, Semigroups and Other Algebraic Structures

Nik Ruskuc

nik@mcs.st-and.ac.uk

School of Mathematics and Statistics, University of St Andrews

York, 26 November 2008

University of St Andrews

Preview: $1 + 1 = 2$

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Instead of an Introduction: $d(A_n)$, $d(S_n)$

Definition

The rank of an algebraic structure is the smallest number of generators needed to generate A; it is denoted d(A).

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Facts

$d(A_5^{19}) = 2$, $d(A_5^{20}) = 3$.

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$$A \xleftarrow{\pi_A : (a,b) \longmapsto a} A \times B \xrightarrow{\pi_B : (a,b) \longmapsto b} B$$

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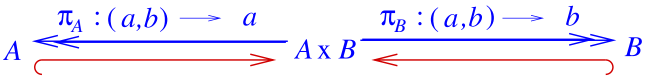

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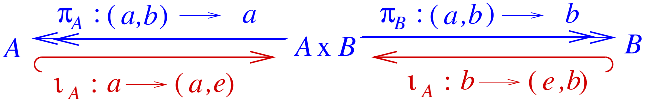

$\pi_A : (a,b) \longrightarrow a$
$\pi_B : (a,b) \longrightarrow b$
A
$A \times B$
B
$\iota_A : a \longrightarrow (a,e)$
$\iota_A : b \longrightarrow (e,b)$

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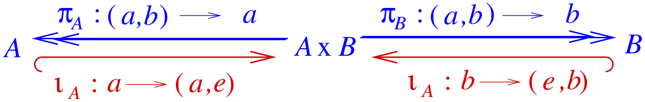

Provided e is an idempotent

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Remark

This works for monoids.

Growth of Direct Powers

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Corollary

For any monoids M, N we have

$$\max(d(M), d(N)) \leq d(M \times N) \leq d(M) + d(N).$$

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For a monoid M we have

$$d(M) \leq d(M^n) \leq nd(M).$$

Growth Sequences: Finite Groups

J. Wiegold (with J.S. Wilson, D. Meier, A.G.R. Stewart, A. Efranian), 1974–1995.

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If a perfect group G has no finite (non-trivial) images, is $\mathbf{d}(G)$ necessarily eventually constant?

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In the non-monoid case, is $\mathbf{d}(S)$ eventually exponential?

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Does such a monoid exist?

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For every $k \geq 1$ there exist $a_1, \ldots, a_k \in T_{\mathbb{N}}$ such that

$$\underbrace{T_{\mathbb{N}} \times \ldots T_{\mathbb{N}}}_{k} = (a_1, \ldots, a_k) T_{\mathbb{N}}.$$

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Proposition

The monoid R of all recursive functions $\mathbb{N} \to \mathbb{N}$ is finitely generated, and also has the above property.

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Remark

Neither $T_{\mathbb{N}}$ nor R are congruence-free.

Growth: Infinite Semigroups: Work in Progress

St Andrews Algebra and Combinatorics Summer School 2008.

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$P_2 = \langle b_i, c_i \mid b_i c_i = 1,\ b_i c_j = 0\ (i, j = 1, 2,\ i \neq j) \rangle$.

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Theorem

$\mathbf{d}(P_2) = (3, 5, 7, 9, \ldots)$.

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- Do there exist semigroups/monoids of intermediate growth?

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Finite Generation: Semigroups

Theorem (EF Robertson, NR, J. Wiegold)

Let S, T be infinite semigroups. $S \times T$ is finitely generated if and only if

(i) *S and T are finitely generated; and*

(ii) *neither S nor T have indecomposable elements.*

Theorem

Let S, T be semigroups, with S infinite, T finite. $S \times T$ is finitely generated if and only if

(i) *S is finitely generated; and*

(ii) *T has no indecomposable elements.*

Finite Presentability

Theorem

Let M, N be monoids. $M \times N$ is finitely presented if and only if M and N are finitely presented.

Example

$\mathbb{N}$ is finitely presented (in fact free), but $\mathbb{N} \times \mathbb{N}$ is not finitely presented.

Question

Will $S \times T$ be finitely presented provided S and T are finitely presented and $S \times T$ is finitely generated?

Critical Pairs and Stability

S – a semigroup; $\langle A|R\rangle$ a finite presentation for it.

Fact

Two words u, v over A are equal in S if and only if there is a sequences of applications of relations from R (a deduction) which transforms u into v.

Definition

A pair (u, v) of words is critical if every deduction from u to v contains a word of length smaller than $\min(|u|, |v|)$.

Definition

S is said to be stable if it has no critical pairs.

Remark

The above definition of stability is not constructive.

Stability and Finite Presentability

Theorem (EF Robertson, NR, J Wiegold)

Let S, T be two infinite semigroups. $S \times T$ is finitely presented if and only if

(i) *S and T are (finitely presented) and stable; and*

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Theorem (EF Robertson, NR, J Wiegold)

Let S be an infinite semigroup, and let T be a finite semigroup. $S \times T$ is finitely presented if and only if

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Some Non-Finitely-Presented Examples

Example (Araujo, NR)

The four element semigroup

S	a	b	c	0
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Residual Finiteness: Definition

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An algebraic structure A is residually finite if for any two $a, b \in A$ ($a \neq b$) there is a homomorphism $f : A \to B$, B finite, such that $f(a) \neq f(b)$.

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Equivalently: the intersection of all finite index congruences is trivial.

Residual Finiteness: General, Nice, Boring Theorem?

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Proof

If $e \in A$ is an idempotent, then $B \cong \{e\} \times B \leq A \times B$.

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R. Gray, NR

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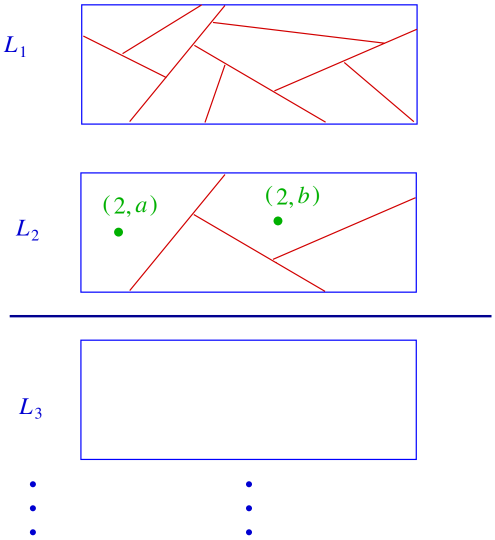
Intersect ρ and λ to obtain a congruence $\sigma = \rho \cap \lambda$ which has finitely many classes, respects levels 1 and 2, and separates $(2, a)$ and $(2, b)$.

Residual Finiteness: Levels of $\mathbb{N} \times S$



Residual Finiteness: Semigroups

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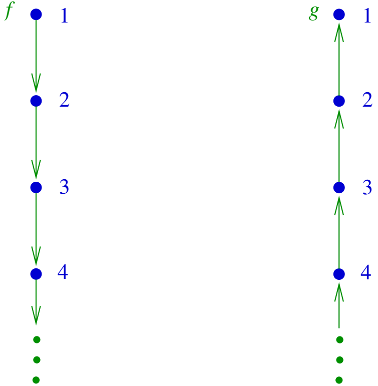
Proof

A semigroup either contains an idempotent or a copy of $\mathbb{N}$:-)

Residual Finiteness: Unary Algebras

Example

Consider two unary operations on $\mathbb{N}$:



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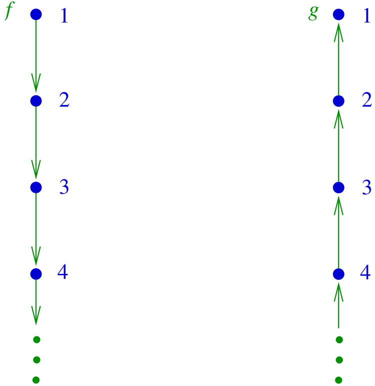

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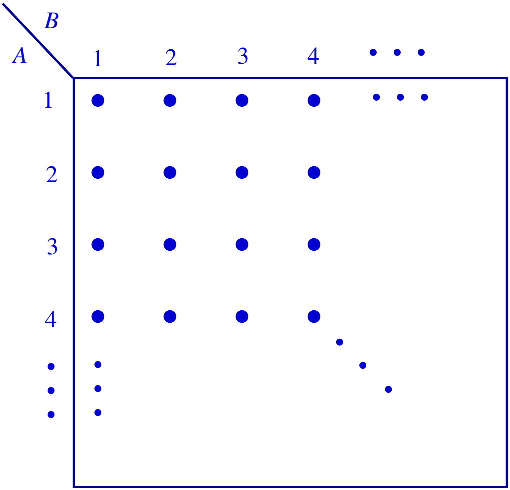

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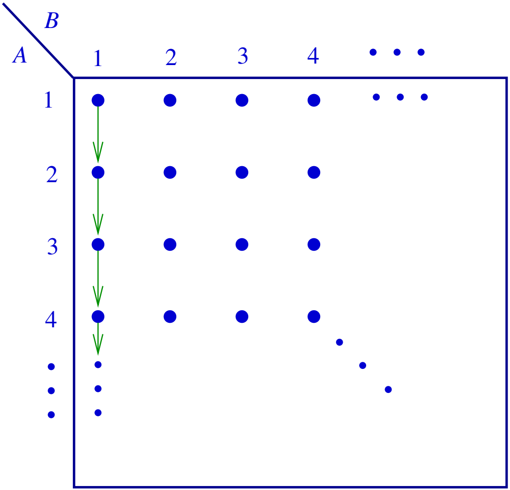

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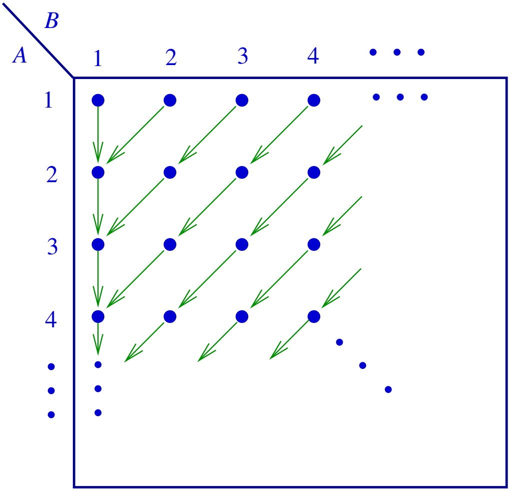

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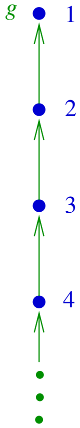

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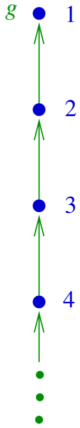

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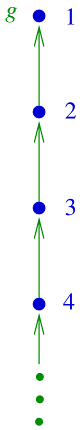

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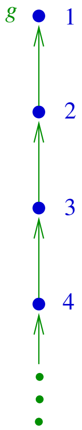

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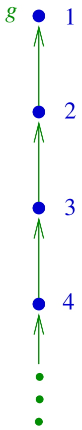

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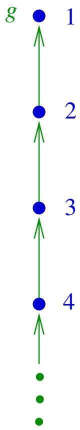

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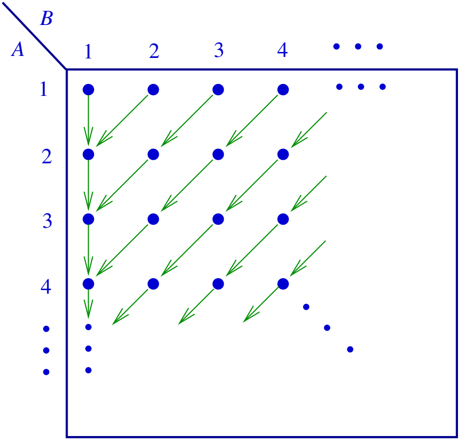

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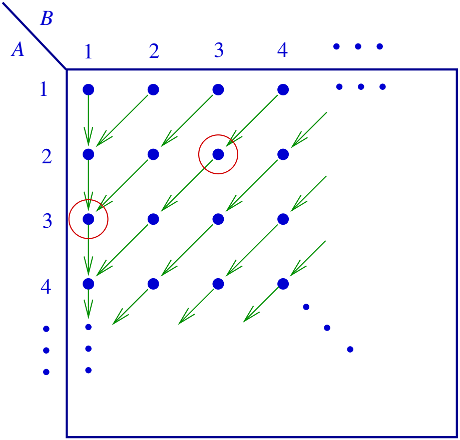

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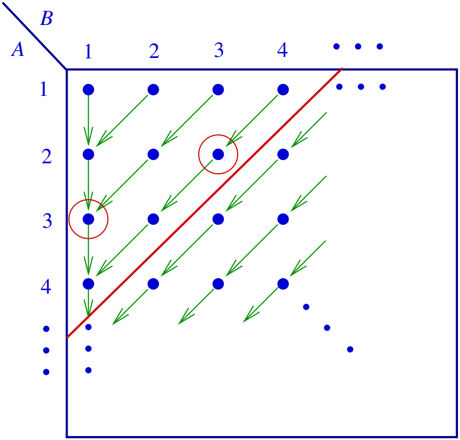

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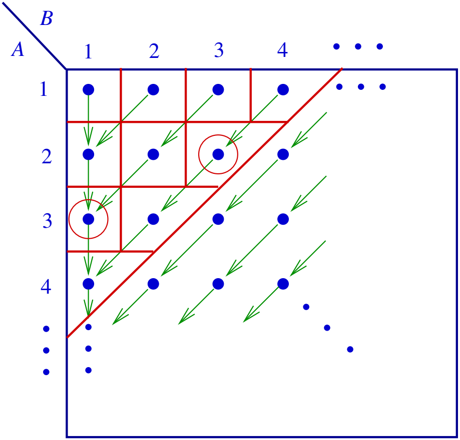

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Other products: wreath product, work in progress with M Quick, M Neunhöffer.

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Investigate the **d**-sequences of S-acts.

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