Diagonal Acts and Applications

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Proof of (b).

Take $S = T_N$, the full transformation monoid.

Let $\alpha, \beta \in T_N$ be defined by $n\alpha = 2n - 1$, $n\beta = 2n$.

Let $\gamma, \delta \in T_N$ be arbitrary.

Define $\mu \in T_N$ by

$$n\mu = \begin{cases} 
  k\gamma & \text{if } n = 2k - 1 \\
  k\delta & \text{if } n = 2k.
\end{cases}$$

Immediate check: $(\alpha \mu, \beta \mu) = (\gamma, \delta)$. 
Definition
Let $S$ be a semigroup. The set $S^n$ on which $S$ acts via

$$(x_1, \ldots, x_n)s = (x_1s, \ldots, x_ns)$$

is called the $(n$-ary, right) diagonal act.

Definition
The diagonal act $S^n$ is finitely generated if there is a finite set $A \subseteq S^n$ such that $S^n = AS$.
It is cyclic if $A$ can be chosen to have size 1.
Diagonal Acts: Examples

Example
If $S$ is any of $T_{\mathbb{N}}, P_{\mathbb{N}}, B_{\mathbb{N}}$ then the diagonal act $S^n$ is finitely generated for all $n$. (This includes $n = \aleph_0$!)

Proposition
No infinite group has a finitely generated diagonal act.

Proof
If $(a, b)s = (c, d)$ then $ab^{-1} = cd^{-1}$.

Proposition (Gallagher 05)
No infinite inverse semigroup has a finitely generated diagonal act.
Diagonal Acts: Examples

Example (Robertson, NR, Thomson 01)
The monoid of all recursive functions $\mathbb{N} \to \mathbb{N}$ is finitely generated and has cyclic diagonal acts.

Example (ibid)
There exists a finitely presented infinite monoid with cyclic diagonal acts.
Lemma
If $S \times S = (a, b)S$ then $S^{2n} = (a_1, \ldots, a_{2^n})S$ where
\{a_1, \ldots, a_{2^n}\} = \{a, b\}^n$.

Sketch of Proof
\((n = 2)\)
\{a, b\}^2 = \{aa, ba, ab, bb\}.
Suppose we are given \((a_1, a_2, a_3, a_4) \in S^4\).
Find \(s_1, s_2, s \in S\) so that:
\((a, b)s_1 = (a_1, a_2),\ (a, b)s_2 = (a_3, a_4),\ (a, b)s = (s_1, s_2)\).

Then
\((aa, ba, ab, bb)s = (as_1, bs_1, as_2, bs_2) = (a_1, a_2, a_3, a_4)\).
Corollary

If the diagonal act $S^n$ is cyclic for some $n \geq 2$ then $S^n$ is cyclic for all $n \geq 2$.

Problem

Does $S^2$ cyclic imply that $S^{\mathbb{N}_0}$ is cyclic?

Problem

Does $S^2$ finitely generated imply $S^n$ finitely generated for all $n \geq 2$?
Applications

- Wreath products (Robertson, NR, Thomson);
- Finitary power semigroups (Robertson, Thomson, Gallagher, NR);
- Ranks of direct powers (Neunhoeffer, Quick, NR).
Definition
Let $S$ be a semigroup. The finitary power semigroup of $S$ (denoted $P_f(S)$) consists of all finite subsets of $S$ under multiplication $A \cdot B = \{ab : a \in A, b \in B\}$.

Question
Can $P_f(S)$ be finitely generated for any infinite semigroup $S$?

Theorem
No, if $S$ is a group (Gallagher, NR, 2007), or inverse semigroup with an infinite subgroup (Gallagher).
Theorem
If $S$ is finitely generated and the diagonal act $S \times S$ is cyclic then $P_f(S)$ is finitely generated.

Proof
Suppose $S = \langle A \rangle$ and $S \times S = (a, b)S$.
Recall $S^{2n} = (a_1, \ldots, a_{2^n})S$, where $\{a_1, \ldots, a_{2^n}\} = \{a, b\}^n$.
For $s \in S$, let $\overline{s} = \{s\}$; clearly $S \cong \overline{S} \leq P_f(S)$.
$P_f(S) = \langle \{a, b\}\overline{S} = \langle \{a, b\}, \overline{A} \rangle$.

Problem
Can $P_f(S)$ be finitely presented for any infinite semigroup $S$?
Growth Sequences of Direct Powers

Definition
\(d(S)\) = the smallest number of generators needed to generate \(S\).

Definition
\(d(S) = (d(S), d(S^2), d(S^3), \ldots)\).

Warning
From now on \(S^n\) may stand for either the \(n\)-ary diagonal act of \(S\) or for the \(n\)th direct power of \(S\)! In the above definition it is the latter.

Example
\(d(C_2) = (1, 2, 3, 4, 5, \ldots)\).

Example (Hall 1936)
\(d(A_5) = (2, 2, 2, \ldots, 2, 3, \ldots)\).

Example (Hall 1936)
\(d(A_5) = (2, 2, 2, \ldots2, 3, \ldots, 3, 4, \ldots)\)
Growth: Finite Groups and Semigroups

Wiegold and various co-authors, 1974–1995.

Theorem

Let $G$ be a finite group.

- If $G$ is perfect then $d(G)$ is logarithmic.
- If $G$ is non-perfect then $d(G)$ is eventually linear.

Theorem

Let $S$ be a finite semigroup.

- If $S$ is a monoid then $d(S)$ is eventually linear.
- If $S$ is not a monoid then $d(S)$ is asymptotically exponential.

Problem

In the last case, is $d(S)$ in fact eventually exponential?
Growth: Infinite Groups

Wiegold and various co-authors, 1974–1995.

**Theorem**

Let $G$ be an infinite group.

- If $G$ is simple then $d(G)$ is eventually constant.
- If $G$ is perfect, non-simple then $d(G)$ is either logarithmic or eventually constant.
- If $G$ is non-perfect then $d(G)$ is eventually linear.
Theorem

Let $S$ be a finitely generated semigroup with a cyclic diagonal act. Then $d(S)$ is eventually constant.

Proof

Recall: d.a. $S^2$ cyclic $\Rightarrow$ d.a. $S^k$ cyclic for all $k$.

Suppose $S = \langle A \rangle$ and $S^k = (a_1, \ldots, a_k)S$.

For $s \in S$, let $\bar{s} = (s, \ldots, s) \in S^k$; clearly $\bar{S} \cong S$.

$S^k = (a_1, \ldots, a_k)\bar{S} = \langle (a_1, \ldots, a_k), \bar{A} \rangle$.

Hence $d(S^k) \leq d(S) + 1$.

Problem

If $S$ is a finitely generated congruence-free semigroup, is $d(S)$ eventually constant?