UNIVERSITY OF ST ANDREWS
School of Mathematics and Statistics
MT4603 Groups: Tutorial 2.

1. Let $n > 1$. Prove that $\{\sigma \in S_n : n\sigma = n\}$ is a subgroup of S_n isomorphic to S_{n-1}.

2. Let G be a group and let H and K be two subgroups of G. Prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

3. For an element a of a group G denote by $|a|$ the order of a.

(i) Prove that for any $a, b \in G$ we have $|a| = |a^{-1}|$, $|a| = |b^{-1}ab|$, $|ab| = |ba|$.

(ii) Prove that for any two $a, b \in G$ of finite order such that $ab = ba$ we have $|ab| \leq \text{l.c.m.}(|a|, |b|)$. Conclude that the set of all elements of finite order in an abelian group is a subgroup.

(iii) Prove that the order of an r-cycle is r. If a permutation $\sigma \in S_n$ is decomposed into a product of disjoint cycles as $\sigma = \gamma_1\gamma_2 \dots \gamma_k$, where the length of γ_i is r_i, then prove that $|\sigma| = \text{l.c.m}(r_1, r_2, \dots, r_k)$.

(iv) Prove that the matrices $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ from $GL(2, \mathbb{Q})$ have orders 4 and 3 respectively, but that the matrix AB does not have finite order.

4. List the elements of the dihedral group D_4 and their orders. Is D_4 isomorphic to $\mathbb{Z}_8$? Is D_4 isomorphic to the quaternion group Q_8? Is D_6 isomorphic to A_4? Is D_{12} isomorphic to S_4?

5. Consider the additive group $\mathbb{Q}$ of rationals. Prove that if $X \subseteq \mathbb{Q}$ is a finite set then the subgroup $\langle X \rangle$ is contained in a cyclic subgroup of $\mathbb{Q}$. Conclude that $\langle X \rangle$ is cyclic. Is $\mathbb{Q}$ cyclic? Conclude that $\mathbb{Q}$ cannot be generated by a finite set.