

**UNIVERSITY OF ST ANDREWS**  
**School of Mathematics and Statistics**  
**MT4603 Groups: Tutorial 4.**

1. Prove that if  $N$  and  $K$  are normal subgroups of a group  $G$  then so is  $NK = \{nk : n \in N, k \in K\}$ .
2. Let  $N$  be the subgroup of  $S_4$  generated by  $(1\ 2)(3\ 4)$  and  $(1\ 3)(2\ 4)$ .
  - (i) Find the elements of  $N$  and prove that  $N$  is normal in  $S_4$ .
  - (ii) List the elements of the quotient  $S_4/N$ . Is  $S_4/N$  cyclic? Is it abelian?
  - (iii) Find a subgroup  $M$  of  $N$  of order 2. Prove that  $M$  is normal in  $N$  but that  $M$  is not normal in  $S_4$ .
  - (iv) Prove that the dihedral group  $D_4$  contains two normal subgroups  $K$  and  $L$  such that  $D_4/K \cong D_4/L$  but  $K \not\cong L$ .
3. Prove that a subgroup of index 2 is normal.
4. Let  $H$  be the subgroup  $H = \{\text{id}, (1\ 2)\}$  of  $S_3$ . Show that

$$(2\ 3)H = (1\ 2\ 3)H \text{ and } (1\ 3\ 2)H = (1\ 3)H$$

but that

$$[(2\ 3)(1\ 3\ 2)]H \neq [(1\ 2\ 3)(1\ 3)]H.$$

5. Find the orders of the elements of  $Q_8$ . Find all the subgroups of  $Q_8$ . Prove that they are all normal in  $Q_8$ . Determine, up to isomorphism, all homomorphic images of  $Q_8$ .