UNIVERSITY OF ST ANDREWS
School of Mathematics and Statistics
MT4603 Groups: Tutorial 5.

1. A permutation group $G \leq S_X$ is said to be *transitive* if for all $i, j \in X$ there exists $\sigma \in G$ such that $i\sigma = j$. Prove that every group is isomorphic to a transitive subgroup of some symmetric group. What is the smallest n such that $\mathbb{Z}_6$ is isomorphic to a transitive subgroup of S_n? What is the smallest n such that $\mathbb{Z}_6$ is isomorphic to a subgroup of S_n?

2. Let $n \geq 1$ be arbitrary. For an arbitrary permutation $\sigma \in S_n$ define a permutation $\overline{\sigma} \in S_{n+2}$ by

$$\overline{\sigma} = \begin{cases} \sigma & \text{if } \sigma \text{ is even} \\ \sigma(n+1\ n+2) & \text{if } \sigma \text{ is odd.} \end{cases}$$

Define a mapping $f : S_n \longrightarrow S_{n+2}$ by $\sigma f = \overline{\sigma}$. Prove that f is a monomorphism (i.e. an injective homomorphism), and that $\mathrm{im} f \subseteq A_{n+2}$. Conclude that every finite group is isomorphic to a subgroup of an alternating group.

3. A homomorphism ρ from a group G to $GL(n, \mathbb{F})$ is called a *representation of G over* $\mathbb{F}$, with degree n.
Let G be $D_4 =< \alpha, \beta : \alpha^4 = \beta^2 = e, \beta^{-1}\alpha\beta = \alpha^{-1} >$.
Define $A, B \in GL(2, \mathbb{F})$ by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(i)Show that $A^4 = B^2 = I$ and $B^{-1}AB = A^{-1}$.
(ii)Prove that the function $\rho : \alpha^i\beta^j \mapsto A^iB^j (0 \leq i \leq 3, 0 \leq j \leq 1)$ is a representation of D_4 of degree 2.
A representation is called *faithful* if it is injective.
(iii) Show that ρ is a faithful representation.
(iv) Give an example of a faithful representation of D_4 of degree 3.

4. Let H and K be two groups, and let $\overline{H} = \{(h, e_K) : h \in H\}$. We proved in lectures that $\overline{H} \trianglelefteq H \times K$. Prove that $(H \times K)/\overline{H} \cong K$.