

UNIVERSITY OF ST ANDREWS
School of Mathematics and Statistics
MT4603 Groups: Tutorial 6.

1. Prove that the direct product $H \times K$ is abelian if and only if both H and K are abelian. Prove that the order of an element $(h, k) \in H \times K$ is equal to $\text{lcm}(|h|, |k|)$. Give examples of at least three non-abelian groups of order 24, and prove that they are indeed non-isomorphic.
2. Prove that the quaternion group Q_8 is directly indecomposable.
3. Let p be a prime, let $n \geq 1$, and consider the cyclic group \mathbb{Z}_{p^n} . Prove that every non-trivial subgroup of \mathbb{Z}_{p^n} contains the subgroup $H = \langle p^{n-1} \rangle$. Conclude that \mathbb{Z}_{p^n} is directly indecomposable.
4. List all non-isomorphic abelian groups of order 1000.
5. Prove that a finite abelian group that is not cyclic contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p . Prove that no cyclic group contains such a subgroup.
6. Prove that the cyclic group \mathbb{Z}_{p^n} contains a subgroup of order p^i for every i ($0 \leq i \leq n$). Prove that for every abelian group G and every m dividing the order of G , there is a subgroup of G of order m . (Hint: induction on $|G|$.)