UNIVERSITY OF ST ANDREWS
School of Mathematics and Statistics
MT4603 Groups: Tutorial 7.

1. Let $n \geq 5$. Assume that N is a proper normal subgroup of S_n.

(i) Show that N contains at least one non-identity even permutation.

(ii) Show that $N \cap A_n \trianglelefteq A_n$. Conclude that $A_n \subseteq N$.

(iii) Prove that the only proper normal subgroup of S_n is A_n.

(iv) Prove that S_n is directly indecomposable.

2. Prove that if $f : G \longrightarrow H$ is a non-trivial homomorphism (i.e. f does not send everything into e_H) and if G is simple, then f is a monomorphism. (Hint: prove that $kerf = \{e_G\}$.)

3. Consider the alternating group A_4. We have the two series:

(I) $e \leq \langle (1\ 2\ 3) \rangle \leq A_4$

(II) $e \leq \langle (1\ 3)(2\ 4) \rangle \leq \langle (1\ 3)(2\ 4), (1\ 2)(3\ 4) \rangle \leq A_4$.

(i) Is (I) a subnormal series? A composition series?

(ii) Is (II) a subnormal series? A composition series?

(iii) Find the composition factors of A_4.

4. Prove that the centre of the symmetric group S_n, $n \geq 3$, is trivial. Prove that the centre of the dihedral group D_n is trivial if n is odd, and is equal to $\{\mathrm{id}, \alpha^{n/2}\}$ if n is even (here $\alpha = (1\ 2\ \ldots\ n)$). Conclude that $S_n \ncong D_{n!/2}$ for $n \geq 4$.

5. Let G be a group. Prove that the inverse of the commutator $[a, b]$ of two elements $a, b \in G$ is again a commutator. Prove that a conjugate of a commutator is also a commutator.

6. Prove that $(G \times H)' = G' \times H'$. (Hint: Prove that every generator of each of these groups belongs to the other.)